

GENERIC VARIABLES IN ACYCLIC CLUSTER ALGEBRAS AND BASES IN AFFINE CLUSTER ALGEBRAS

G. DUPONT

ABSTRACT. Let Q be a finite quiver without oriented cycles and $\mathcal{A}(Q)$ be the coefficient-free cluster algebra with initial seed (Q, \mathbf{u}) . Using the Caldero-Chapoton map, we introduce and investigate a family of generic variables in $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ containing the cluster monomials of $\mathcal{A}(Q)$. The aim of these generic variables is to give an explicit new method for constructing \mathbb{Z} -bases in the cluster algebra $\mathcal{A}(Q)$.

If Q is an affine quiver with minimal imaginary root δ , we investigate differences between cluster characters associated to indecomposable representations of dimension vector δ . We define the notion of *difference property* which gives an explicit description of these differences. We prove in particular that this property holds for quivers of affine type \tilde{A} . When Q satisfies the difference property, we prove that generic variables span the cluster algebra $\mathcal{A}(Q)$. If $\mathcal{A}(Q)$ satisfies some gradability condition, we prove that generic variables are linearly independent over \mathbb{Z} in $\mathcal{A}(Q)$. In particular, this implies that generic variables form a \mathbb{Z} -basis in a cluster algebra associated to an affine quiver of type \tilde{A} .

CONTENTS

Introduction	1
1. Background and main results	3
2. Affine cluster algebras and cluster categories	10
3. Generic variables	15
4. Reflection functors and Caldero-Chapoton map	36
5. Examples of semicanonical bases	52
6. Conjectures and questions	60
References	62

INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky in a series of papers [FZ02, FZ03, BFZ05, FZ07] in order to design a general framework for understanding total positivity in algebraic groups and canonical bases in quantum groups. They turned out to be related to various subjects in mathematics like combinatorics, Lie theory, representation theory, Teichmüller theory and many other topics.

A cluster algebra is a commutative algebra generated by indeterminates over \mathbb{Q} called *cluster variables*. They are gathered into sets of fixed cardinality called *clusters*. The initial data for constructing a (symmetric coefficient-free) cluster algebra is a *seed*, that is, a pair (B, \mathbf{u}) where $B \in M_q(\mathbb{Z})$ is an anti-symmetric matrix and $\mathbf{u} = (u_1, \dots, u_q)$ is a q -tuple of indeterminates over \mathbb{Q} . The cluster variables are defined inductively by a process called *mutation*. The cluster algebra associated to a seed (B, \mathbf{u}) is denoted by $\mathcal{A}(B)$, it is a \mathbb{Z} -subalgebra of the ring $\mathbb{Z}[u_1^{\pm 1}, \dots, u_q^{\pm 1}]$ of Laurent polynomials in \mathbf{u} .

Viewing anti-symmetric matrices as incidence matrices of quivers, it is possible to define the cluster algebra $\mathcal{A}(Q)$ from a seed (Q, \mathbf{u}) where Q is a quiver. Namely, $\mathcal{A}(Q)$ is the cluster algebra with initial seed (B, \mathbf{u}) where B is the incidence matrix of Q . If Q contains no oriented cycles, it is called *acyclic* and $\mathcal{A}(Q)$ is called an *acyclic* cluster algebra. Initiated in [MRZ03], formally defined in [BMR⁺06] and later developed in various papers (see [BMR07, BMR08, BMR09, CCS06b, CCS06a, CC06, CK08, CK06] for example), the *cluster category* of an acyclic quiver Q gives a fruitful categorification of an acyclic cluster algebra $\mathcal{A}(Q)$. Another categorification, using preprojective algebras is developed independently by Geiss, Leclerc and Schröer (see [GLS05, GLS07, GLS08] for example). For a wide overview concerning categorifications of acyclic cluster algebras, one can refer to [Kel08].

The investigation of \mathbb{Z} -bases in cluster algebras, whether it is very fundamental in the theory, is still widely open. A first idea would be to consider particular monomials in cluster variables in order to constitute a \mathbb{Z} -basis of $\mathcal{A}(Q)$. A privileged choice is given by the *cluster monomials*, that is, monomials in cluster variables taken in a same cluster. It is proved in [CK08] (see also [SZ04] for rank two) that cluster monomials constitute indeed a basis if Q is a Dynkin quiver.

In [SZ04], the authors investigated all cluster algebras associated to matrices of rank 2 of Dynkin or affine type. It turned out that if the matrix B is not of Dynkin type, the cluster monomials are not enough to generate $\mathcal{A}(B)$ as a \mathbb{Z} -module and one has to add somehow an “imaginary” part to the set of cluster monomials. For these cluster algebras of rank 2, they managed to obtain a \mathbb{Z} -basis called *canonical basis*, characterized by a certain positivity property. Nevertheless, at this time there are no similar results for cluster algebras of higher ranks.

Later, in [CZ06], Caldero and Zelevinsky investigated another basis arising naturally from the representation theory of the quiver Q when Q is the Kronecker quiver. They called it the *semicanonical basis* of $\mathcal{A}(Q)$. It has no longer the positivity property of the canonical basis but it appeared to us that the definition of Caldero-Zelevinsky’s basis, using the AR-quiver approach to cluster variables, could be generalized (see also [Cer08] for results about concerning rank three cluster algebras).

In [GLS08], the authors give a very general definition of bases in cluster algebras using Lusztig’s dual semicanonical basis. Their methods consist in realizing certain cluster algebras as subalgebras of the graded dual of the enveloping algebra $U(\mathfrak{n})$ where \mathfrak{n} is the maximal nilpotent subalgebra of the Lie-Kac-Moody algebra \mathfrak{g} associated to Q . Specializing coefficients, they construct a basis in the acyclic cluster algebra $\mathcal{A}(Q)$ using the dual \mathcal{S}^* of Lusztig’s semicanonical basis. By definition, the elements of \mathcal{S}^* are generic constructible functions parametrized by irreducible components of the nilpotent varieties of modules over the preprojective algebra.

The idea of this article is to define generic values in an acyclic cluster algebra, giving an analogue in the context of cluster categories to the generic functions of Lusztig’s dual semicanonical basis. This enables to carry out the methods used in [CK08] for Dynkin quivers to quivers of affine types in order to define a \mathbb{Z} -basis. The interest of this method is that it provides a completely explicit realization of the generic values in an acyclic cluster algebra.

The main tool for constructing these generic variables is the so-called *Caldero-Chapoton map*. Introduced in [CC06], the Caldero-Chapoton map is an explicit map from the set of objects in the cluster category \mathcal{C}_Q to the ring of Laurent polynomials in \mathbf{u} containing the acyclic cluster algebra $\mathcal{A}(Q)$. Among various interesting properties, the Caldero-Chapoton map turns out to be a constructible function, admitting generic values. It turns out that cluster monomials naturally appear as generic values. Regarding the works of Geiss, Leclerc and Schröer, it is reasonable

to think that generic values for the Caldero-Chapoton map form a good candidate for being a \mathbb{Z} -basis in the considered cluster algebra. The article is devoted to prove that this is indeed the case when Q is an affine quiver satisfying a property called *difference property*. In particular, this proves that generic variables form indeed a \mathbb{Z} -basis when Q is a quiver of affine type \tilde{A} .

Because of the similarity with Lusztig's dual semicanonical basis, it seemed to be relevant to call our basis the *semicanonical basis* of the cluster algebra. However, this might look a bit confusing because, even if the terminology is the same, our semicanonical basis does not coincide with Caldero-Zelevinsky's semicanonical basis explicited in [CZ06] for the Kronecker quiver. We will investigate in subsection 5.2 the differences between known bases in this case. We will prove in particular that the Caldero-Zelevinsky's basis, the canonical basis and the semicanonical basis can be obtained from each other only by a locally unipotent base change.

The article is organized as follows. Section 1 presents the used material and the main results of this paper. In section 2, we give the essential background concerning cluster categories and cluster algebras associated to affine quivers. Section 3 is the heart of our study. We introduce the so-called *generic variables* and study some of their properties for acyclic quivers. We then investigate their properties for affine quivers and prove that they form a generating set in cluster algebras $\mathcal{A}(Q)$ associated to affine quivers Q satisfying a certain *difference property*. In section 4, we prove that the Caldero-Chapoton map is compatible with Zhu's extended BGP-reflection-functors. Considering interactions of reflection functors and generic bases, this allows to prove that under some gradability condition on $\mathcal{A}(Q)$, generic variables are linearly independent in $\mathcal{A}(Q)$. In section 5, we give explicit computations of generic variables for Dynkin quivers and affine quivers of rank lesser than four. Finally in section 6, we give some conjectures and questions.

1. BACKGROUND AND MAIN RESULTS

In this article, k denotes the field \mathbb{C} of complex numbers.

1.1. The cluster category. Let Q be a quiver, we denote by Q_0 the set of vertices, Q_1 the set of arrows and for any arrow $\alpha : i \rightarrow j$ we denote by $s(\alpha) = i$ the *source* of α and by $t(\alpha) = j$ the *tail* of α . All the considered quivers will be connected (the underlying diagram is connected) and finite (Q_0 and Q_1 are finite sets). A quiver Q is called *acyclic* if Q does not contain any oriented cycle.

Let Q be an acyclic quiver, $\Phi(Q)$ its root system, $\Phi_{\geq 0}(Q)$ its positive root system and $\Pi(Q) = \{\alpha_i, i \in Q_0\}$ the set of simple roots of Q . As usual, we write

$$\Phi_{\geq -1}(Q) = \Phi_{\geq 0}(Q) \sqcup -\Pi(Q)$$

the set of *almost positive roots*. We will denote by $(-, -)$ the Tits form of Q . We identify the root lattice with \mathbb{Z}^{Q_0} by sending α_i to the i -th vector of the canonical basis of \mathbb{Z}^{Q_0} .

Let $\text{rep}(Q)$ denote the category of finite dimensional representations of Q . If M is a representation of Q , then for any $i \in Q_0$, $M(i)$ denotes the underlying vector space at vertex i and for any $\alpha : i \rightarrow j$ in Q_1 , $M(\alpha) : M(i) \rightarrow M(j)$ denotes the corresponding linear map. The *dimension vector* of M is the element in \mathbb{N}^{Q_0} defined by

$$\dim M = (\dim M(i))_{i \in Q_0}.$$

Let kQ be the path algebra over Q and $kQ\text{-mod}$ be the category of finite dimensional left- kQ -modules. It is known that $kQ\text{-mod}$ is equivalent to the category $\text{rep}(Q)$, we will then often abuse the terminology by identifying modules and representations. For any vertex $i \in Q_0$, we denote by S_i the simple module associated to i , by P_i its projective cover and by I_i its injective hull.

For any dimension vector \mathbf{d} , $\text{rep}(Q, \mathbf{d})$ denotes the set of representations of Q with dimension vector \mathbf{d} . This is an affine variety isomorphic to

$$\text{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} k^{d_s(\alpha)} \times k^{d_t(\alpha)}.$$

In particular $\text{rep}(Q, \mathbf{d})$ is an irreducible variety. The algebraic group

$$G_{\mathbf{d}} = \prod_{i \in Q_0} GL(k, d_i)$$

acts on $\text{rep}(Q, \mathbf{d})$ and the $G_{\mathbf{d}}$ orbits in $\text{rep}(Q, \mathbf{d})$ coincide with the isoclasses of kQ -modules.

The Grothendieck group $K_0(kQ)$ of kQ -mod is the free abelian group over the isoclasses of kQ -modules modulo the relations $X + Y = E$ for any short exact sequence $0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0$. The dimension vector \mathbf{dim} induces an isomorphism of abelian groups

$$\mathbf{dim} : K_0(kQ) \xrightarrow{\sim} \mathbb{Z}^{Q_0}$$

sending the isoclass of the simple module S_i to the simple root α_i for any $i \in Q_0$.

As kQ is hereditary, the Euler form on kQ -mod is given by

$$\langle M, N \rangle = \dim \text{Hom}_{kQ}(M, N) - \dim \text{Ext}_{kQ}^1(M, N).$$

It is well defined on the Grothendieck group and for any two kQ -modules M, N , we have

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle = (\mathbf{dim} M, \mathbf{dim} N).$$

We denote by τ the Auslander-Reiten translation on kQ -mod. An indecomposable module will be called *preprojective* if it is in the τ -orbit of a projective module. It will be called *preinjective* if it is in the τ -orbit of an injective module. It will be called *regular* if it is neither preprojective nor preinjective. An arbitrary module is called preprojective (resp. preinjective, regular) if all its indecomposable direct summands are preprojective (resp. preinjective, regular). We denote by $\mathcal{P}(Q)$ (resp. $\mathcal{I}(Q)$, $\mathcal{R}(Q)$) the full subcategory of preprojective (resp. preinjective, regular) modules. If there is no possible confusion, we will omit the reference to Q .

We denote by $D^b(kQ)$ the bounded derived category of kQ -mod with translation τ and shift functor $[1]$. A complex concentrated in degree zero will still be called a module. $D^b(kQ)$ is a triangulated category and the functor $G = \tau^{-1}[1]$ is an auto-equivalence of $D^b(kQ)$. The orbit category $\mathcal{C}_Q = D^b(kQ)/G$, introduced in [BMR⁺06], is called the *cluster category of Q* . It is proved in [Kel05] that \mathcal{C}_Q is a triangulated category and that the canonical functor $D^b(kQ) \rightarrow \mathcal{C}_Q$ is a triangle functor. The image of an object M in $D^b(kQ)$ under this canonical functor will still be denoted by M .

The cluster category \mathcal{C}_Q is a k -linear Krull-Schmidt category whose indecomposable objects are indecomposable modules and shifts of indecomposable projective modules. Namely, if we denote by $\text{ind-}\mathcal{K}$ the family of indecomposable objects in a Krull-Schmidt category \mathcal{K} , we have:

$$\text{ind-}\mathcal{C}_Q = \text{ind-}kQ\text{-mod} \sqcup \{P_i[1] : i \in Q_0\}$$

Every object M in \mathcal{C}_Q has thus a unique decomposition $M = M_0 \oplus P_M[1]$ where M_0 is a module and P_M is a projective module. The homological functor $H^0 = \text{Hom}_{\mathcal{C}_Q}(kQ, -) : \mathcal{C}_Q \rightarrow kQ\text{-mod}$ allows to recover the module part of an object $M \in \mathcal{C}_Q$. Thus, we will always write

$$M = H^0(M) \oplus P_M[1]$$

the decomposition of an object in \mathcal{C}_Q into a direct sum of a module and the shift of a projective module.

The cluster category \mathcal{C}_Q is a 2-Calabi-Yau category, it means that for any two objects $M, N \in \mathcal{C}_Q$, there is a duality

$$\mathrm{Ext}_{\mathcal{C}_Q}^1(M, N) \simeq D\mathrm{Ext}_{\mathcal{C}_Q}^1(N, M)$$

where $D = \mathrm{Hom}_k(-, k)$ is the standard duality. Moreover, if M and N are modules, there is a precise description:

$$\mathrm{Ext}_{\mathcal{C}_Q}^1(M, N) = \mathrm{Ext}_{kQ}^1(M, N) \oplus D\mathrm{Ext}_{kQ}^1(N, M)$$

In the Auslander-Reiten quiver of a cluster category, indecomposable preprojective and indecomposable injective are in the same component. This component is called *transjective* and is denoted by $\mathcal{PT}(Q)$.

1.2. Affine quivers. An *affine quiver* is an acyclic quiver whose underlying diagram is an extended Dynkin diagram. The representation theory of such quivers is deeply studied and the reader can for example refer to [DR76, Rin84, CB92]. We recall some useful background concerning representation theory of affine quivers. In this subsection we always assume that Q is an affine quiver.

We set

$$\Phi^{\mathrm{re}}(Q) = \{\alpha \in \Phi(Q) : (\alpha, \alpha) = 1\} \text{ and } \Phi_{\geq 0}^{\mathrm{re}}(Q) = \Phi^{\mathrm{re}}(Q) \cap \Phi_{\geq 0}(Q)$$

the sets of *real roots* and *positive real roots*,

$$\Phi^{\mathrm{im}}(Q) = \{\alpha \in \Phi(Q) : (\alpha, \alpha) = 0\} \text{ and } \Phi_{\geq 0}^{\mathrm{im}}(Q) = \Phi^{\mathrm{im}}(Q) \cap \Phi_{\geq 0}(Q).$$

the sets of *imaginary roots* and *positive imaginary roots*. It is known that we have the following decomposition:

$$\Phi(Q) = \Phi^{\mathrm{re}}(Q) \sqcup \Phi^{\mathrm{im}}(Q) \text{ and } \Phi_{\geq 0}(Q) = \Phi_{\geq 0}^{\mathrm{re}}(Q) \sqcup \Phi_{\geq 0}^{\mathrm{im}}(Q).$$

Moreover, there exists a unique positive imaginary root δ such that

$$\Phi^{\mathrm{im}}(Q) = \mathbb{Z}\delta$$

and this root δ is called the *minimal imaginary root of Q* . Note that δ is a *sincere root*, it means that $\delta_i \neq 0$ for every $i \in Q_0$.

A positive root α is called a *Schur root* if there exists a (necessarily indecomposable) representation $M \in \mathrm{rep}(Q, \alpha)$ such that $\mathrm{End}_{kQ}(M) \simeq k$. Such a representation is called a *Schur representation*. According to Kac's theorem, there exists an indecomposable representation in $\mathrm{rep}(Q, \alpha)$ if and only if α is a positive root. Moreover, if α is a real root, there exists a unique indecomposable representation of dimension vector α . If in addition α is a Schur root, then M is *rigid*, that is, $\mathrm{Ext}_{kQ}^1(M, M) = 0$. If α is a positive imaginary root, then, the set of indecomposable representations of dimension vector α is a $\mathbb{P}^1(k)$ -family of pairwise non-isomorphic representations. In particular $\mathrm{rep}(Q, \alpha)$ (and thus $\mathrm{rep}(Q)$) contains infinitely many non-isomorphic indecomposable objects.

The existence of a minimal imaginary root provides a very useful linear form $\partial : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ called *defect form*. This form is defined by

$$\partial_\alpha = \langle \delta, \alpha \rangle$$

for any $\alpha \in \mathbb{Z}^{Q_0}$. As the Euler-form is well defined on the Grothendieck group, for any M in $\mathrm{rep}(Q)$, we define the *defect of M* as

$$\partial_M = \langle \delta, \mathbf{dim} M \rangle.$$

The defect allows to characterize whether an indecomposable module is preprojective, preinjective or regular. Namely if M is an indecomposable module, then M is preprojective iff $\partial_M < 0$, M is preinjective iff $\partial_M > 0$ and M is regular iff $\partial_M = 0$.

We denote by $c : K_0(kQ) \rightarrow K_0(kQ)$ the *Coxeter transformation* on $K_0(kQ)$, that is, the \mathbb{Z} -linear transformation induced by the translation on the Grothendieck

group. For any $\alpha, \beta \in \mathbb{Z}^{Q_0}$, we have $\langle \alpha, \beta \rangle = -\langle \beta, c(\alpha) \rangle$. In particular as $c(\delta) = \delta$, we get

$$\langle \alpha, \delta \rangle = -\langle \delta, \alpha \rangle$$

for any $\alpha \in \mathbb{Z}^{Q_0}$.

A *tube* \mathcal{T} is a category isomorphic to the mesh category of the stable translation quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^p)$ where $p \geq 1$ is an integer called *rank* of \mathcal{T} . A tube will be called *homogeneous* if it has rank 1 and it will be called *exceptional* if it is not homogeneous. Abusing the terminology, a quiver of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^p)$ will also be called a tube. It is known that the regular components of the AR quiver of Q form a \mathbb{P}^1 -family of tubes. At most three of the tubes are exceptional. The *tubular type* of Q is the set of ranks of exceptional tubes. For any $\lambda \in \mathbb{P}^1$, we denote by $\mathcal{T}_\lambda(Q)$ the corresponding tube in the AR-quiver of Q . We denote by $\mathbb{P}_0^1(Q)$ the set of $\lambda \in \mathbb{P}^1$ such that \mathcal{T}_λ is homogeneous. If there is no possible confusion, the reference to Q will be omitted.

For a Dynkin quiver Q , one can define an ordering on the indecomposable objects in $kQ\text{-mod}$ with respect to the existence of a morphism from an object to another. For affine quivers such an ordering does not exist. Nevertheless, there is still a result of ordering with respect to the components. If $P \in \mathcal{P}(Q)$, $I \in \mathcal{I}(Q)$ and $R \in \mathcal{R}(Q)$, then

$$\text{Hom}_{kQ}(R, P) \simeq \text{Hom}_{kQ}(I, R) \simeq \text{Hom}_{kQ}(I, P) = 0,$$

and

$$\text{Ext}_{kQ}^1(P, R) \simeq \text{Ext}_{kQ}^1(R, I) \simeq \text{Ext}_{kQ}^1(P, I) = 0.$$

If M and N are two regular indecomposable modules in different tubes, then

$$\text{Hom}_{kQ}(M, N) = 0 \text{ and } \text{Ext}_{kQ}^1(M, N) = 0.$$

Every tube \mathcal{T} is a uniserial abelian category closed under extensions, kernels and cokernels. A regular module M in a tube \mathcal{T} is called *quasi-simple* if it does not contain any proper submodule in \mathcal{T} . In particular, if M is quasi-simple, then it contains no proper regular submodule (these modules are sometimes called *regular simple* in the literature). The *quasi-composition series* of an indecomposable regular module M is a sequence

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i is regular and M_i/M_{i-1} is a quasi-simple module. Such a quasi-composition series is unique. The *quasi-length* $\text{q.l.}(M)$ of M is the integer r , the *quasi-socle* $\text{q.soc}(M)$ of M is M_1 and the *quasi-radical* $\text{q.rad}(M)$ of M is M_{r-1} .

Note that the τ -orbit of any projective or injective module is infinite. The situation is pictured in figure 1.

1.3. The Caldero-Chapoton map. We now return to the case where Q is an arbitrary acyclic quiver. We denote by $\mathcal{A}(Q)$ the coefficient free cluster algebra with initial seed (Q, \mathbf{u}) where $\mathbf{u} = (u_i, i \in Q_0)$ is a set of indeterminates over \mathbb{Q} . It has already been mentioned that the cluster category is a fruitful categorification for the cluster algebra $\mathcal{A}(Q)$. A central object for the ‘decategorification’ in this framework is the so-called Caldero-Chapoton map introduced in [CC06].

Fix M a kQ -module and \mathbf{e} a dimension vector, the *quiver grassmannian* of M of dimension \mathbf{e} is the set

$$\text{Gr}_{\mathbf{e}}(M) = \{N \in \text{rep}(Q, \mathbf{e}) : N \text{ is a subrepresentation of } M\}.$$

This is a closed subset of the standard vector space grassmannian and it is thus a projective variety. We can in particular define its Euler-Poincaré characteristic $\chi(\text{Gr}_{\mathbf{e}}(M))$ with respect to the étale cohomology with compact support. These varieties are of great interest, it is for example proved in [CR08] that $\chi(\text{Gr}_{\mathbf{e}}(M)) > 0$

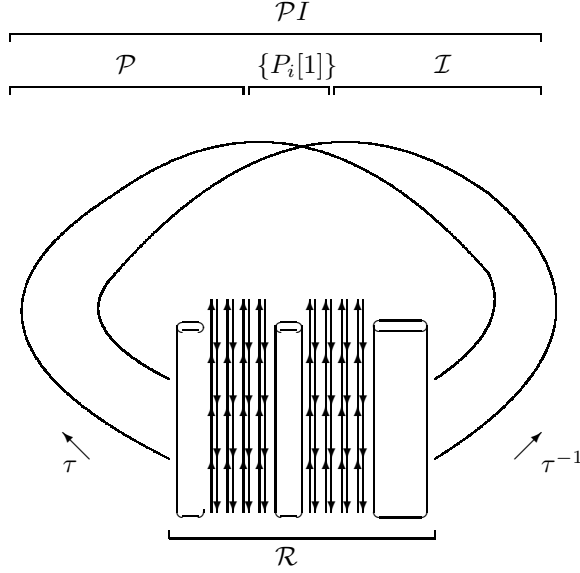


FIGURE 1. The cluster category of an affine quiver

if $\text{Gr}_{\mathbf{e}}(M)$ is not empty which is a very important step towards the proof of the *positivity conjecture* of Fomin and Zelevinsky.

Definition 1.1. The *Caldero-Chapoton map* of an acyclic quiver Q is the map X_Q^Q defined from the set of objects in \mathcal{C}_Q to the ring of Laurent polynomials in the indeterminates $\{u_i, i \in Q_0\}$ by:

(1) If M is an indecomposable kQ -module, then

$$(1) \quad X_M^Q = \sum_{\mathbf{v}} \chi(\text{Gr}_{\mathbf{v}}(M)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \mathbf{dim} M - \mathbf{v} \rangle}$$

(2) If $M = P_i[1]$ is the shift of the projective module associated to $i \in Q_0$, then

$$X_M^Q = u_i$$

(3) For any two objects M, N of \mathcal{C}_Q ,

$$X_{M \oplus N}^Q = X_M^Q X_N^Q$$

For any object M in the cluster category \mathcal{C}_Q , X_M^Q will be called the *generalized variable* associated to M . If there is no possible confusion, we will omit the reference to Q .

Note that the Caldero-Chapoton map satisfies equality (1) for every kQ -module M (see [CC06]). If there is no possible confusion, we will simply write X_M for X_M^Q .

We extend the dimension vector to the objects in the cluster category by setting for any $i \in Q_0$, $\mathbf{dim} P_i[1] = -\alpha_i$ and if $M = \bigoplus_i M_i$ is a decomposition into

indecomposable objects, we set

$$\mathbf{dim} M = \sum_i \mathbf{dim} M_i.$$

Considering a Laurent polynomial $F = P(\mathbf{u}) / \prod_{i \in Q_0} u_i^{d_i}$ such that $P(\mathbf{u})$ is not divisible by any u_i , we define the *denominator vector* $\delta(F)$ of F as the tuple $\mathbf{d} = (d_i)_{i \in Q_0}$. The following theorem will be referred to as the *denominators theorem*:

Theorem 1.2 ([CK08]). *Fix Q an acyclic quiver. Then for any object M in \mathcal{C}_Q , the denominator vector of X_M^Q is $\mathbf{dim} M$.*

The central role of the Caldero-Chapoton map in the ‘decategorification’ is that it allows to realize cluster variables as generalized variables associated to indecomposable rigid objects. Namely, the result is:

Theorem 1.3 ([CK06]). *Let Q be an acyclic quiver. Then*

$$\mathrm{Cl}(Q) = \{M \in \mathrm{Ob}(\mathcal{C}_Q) : M \text{ is indecomposable and rigid}\}.$$

where $\mathrm{Cl}(Q)$ denotes the set of cluster variables in $\mathcal{A}(Q)$.

In [CK08], it turned out that using the properties of the Caldero-Chapoton map, one is able to prove that the set of cluster monomials is a \mathbb{Z} -basis of the \mathbb{Z} -module $\mathcal{A}(Q)$ when Q is a Dynkin quiver. According to theorem 1.3, an equivalent way to state this result is:

Theorem 1.4 ([CK08]). *Let Q be a quiver of Dynkin type. Then the set*

$$\{X_M : M \text{ is rigid in } \mathcal{C}_Q\}$$

is a \mathbb{Z} -basis of the \mathbb{Z} -module $\mathcal{A}(Q)$.

1.4. Main results. The aim of this article is to give a generalization of theorem 1.4 when Q is a quiver of affine type. It already appeared in [SZ04, CZ06] that if Q is not of Dynkin type, then in general, the cluster monomials do not form a generating family of the \mathbb{Z} -module $\mathcal{A}(Q)$. This should be a special case of a general phenomenon for quivers of infinite representation type (see also [Lec03]).

For our purpose, we will define a generalization of the notion of cluster monomial. By theorem 1.3, cluster monomials correspond to generalized variables associated to rigid objects. It is known that the $G_{\mathbf{d}}$ -orbit of a rigid object $M \in \mathrm{rep}(Q, \mathbf{d})$ is open dense in $\mathrm{rep}(Q, \mathbf{d})$. As the Caldero-Chapoton map on $\mathrm{rep}(Q, \mathbf{d})$ is $G_{\mathbf{d}}$ invariant, cluster monomials can be viewed as values of the Caldero-Chapoton map on a dense open subset of $\mathrm{rep}(Q, \mathbf{d})$. The following result, which is a corollary of lemma 3.1, will generalize the notion of cluster monomial with respect to this property.

Corollary 3.3. *Fix Q an acyclic quiver and \mathbf{d} a dimension vector. Then there exists a dense open subset $U_{\mathbf{d}} \subset \mathrm{rep}(Q, \mathbf{d})$ such that the Caldero-Chapoton map is constant over $U_{\mathbf{d}}$. Moreover, if $U'_{\mathbf{d}}$ is another such open subset, the values of the Caldero-Chapoton map are the same on $U_{\mathbf{d}}$ and $U'_{\mathbf{d}}$.*

If $\mathbf{d} \in \mathbb{N}^{Q_0}$, we write $X_{\mathbf{d}}$ the value of the Caldero-Chapoton map on $U_{\mathbf{d}}$. For $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we write

$$[\mathbf{d}]_+ = (\max(d_i, 0))_{i \in Q_0}$$

and we set

$$X_{\mathbf{d}} = X_{[\mathbf{d}]_+} \cdot \prod_{d_i < 0} u_i^{-d_i}$$

the *generic variable of dimension \mathbf{d}* . The following lemma states that generic variables generalize cluster monomials:

Lemma 3.10. *Let Q be an acyclic quiver, fix x a cluster monomial with denominator vector $\delta(x)$. Then*

$$x = X_{\delta(x)}.$$

Equivalently, if M is a rigid object in \mathcal{C}_Q , we have

$$X_M = X_{\dim M}.$$

We set

$$\mathcal{B}'(Q) = \{X_{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{Q_0}\}$$

to be the set of all generic variables. We can explicitly describe the generic variables for an affine quiver. If \mathcal{E} is a set of objects in \mathcal{C}_Q , we denote by $\text{add } \mathcal{E}$ its additive closure, that is, the full subcategory of \mathcal{C}_Q whose objects are direct sums of direct summands of objects in \mathcal{E} . The result is the following:

Proposition 3.18. *Let Q be an affine quiver and $\mathbf{d} \in \mathbb{Z}^{Q_0}$, denote by \mathcal{E} a set of representatives of isoclasses of regular modules M . Then*

$$\mathcal{B}'(Q) = \{\text{cluster monomials}\} \sqcup \left\{ X_{M_{\lambda}^{\oplus n} \oplus E} : E \in \text{add } \mathcal{E} \text{ is rigid}, n \geq 1 \right\}$$

where M_{λ} is any quasi-simple in an homogeneous tube.

We will prove several results for cluster algebras of affine types. The following first result is expected whereas the second one is very surprising. It gives very nice relations between generalized variables associated to indecomposable modules having the same dimension vectors for a cluster algebra of affine type $\tilde{\mathbb{A}}$. For any $\lambda \in \mathbb{P}_0^1$, we denote by M_{λ} the unique indecomposable module of dimension δ belonging to the homogeneous tube \mathcal{T}_{λ} and by $M_{\lambda}^{(n)}$ the unique indecomposable regular module of quasi-length n and quasi-socle M_{λ} . For any quasi-simple module E in an exceptional tube, we write M_E the unique indecomposable module of dimension δ with quasi-socle E .

Lemma 3.14. *Let Q be a quiver of affine type. Then for any $\lambda, \mu \in \mathbb{P}_0^1$ and any $n \geq 1$, we have*

$$X_{M_{\lambda}^{(n)}} = X_{M_{\mu}^{(n)}}$$

Theorem 3.25. *Fix Q a quiver of affine type $\tilde{\mathbb{A}}$, E a regular simple module in an exceptional tube, $\lambda \in \mathbb{P}_0^1$, then*

$$X_{M_E} = X_{M_{\lambda}} + X_{\text{q.rad } M_E / E}$$

We say that an affine quiver Q satisfies the *difference property* if it satisfies the above theorem. The main result of this article is the following theorem:

Theorem 4.21. *Let Q be an affine quiver such that every quiver reflection-equivalent to Q satisfies the difference property. Then $\mathcal{B}'(Q)$ is a \mathbb{Z} -basis for the \mathbb{Z} -module $\mathcal{A}(Q)$ called the semicanonical basis of $\mathcal{A}(Q)$.*

It proves in particular that generic variables form a \mathbb{Z} -basis in an affine cluster algebra of type $\tilde{\mathbb{A}}$.

In subsection 4.1, we will study the interaction between Zhu's extended BGP-reflection functors and the Caldero-Chapoton map. For terminology, the reader can refer to subsection 4.1. An important result is the following:

Proposition 4.6. *Let Q be an affine quiver with at least three vertices. Let i be a sink in Q_0 . Assume that Q and $\sigma_i Q$ satisfy the difference property. Denote by $\Phi_i : \mathcal{A}(Q) \rightarrow \mathcal{A}(\sigma_i Q)$ the canonical isomorphism and by $R_i^+ : \mathcal{C}_Q \rightarrow \mathcal{C}_{\sigma_i Q}$ the extended BGP functor. Then for any object M in \mathcal{C}_Q , we have $\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$.*

We can then deduce the behaviour of the generic variables under reflection functors:

Corollary 4.10. *Let Q be an affine quiver of affine type with at least three vertices. Let i be a sink in Q_0 . Assume that Q and Q' satisfy the difference property. Denote by $\Phi : \mathcal{A}(Q) \rightarrow \mathcal{A}(\sigma_i Q)$ the canonical isomorphism. Then for any $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we have*

$$\Phi(X_{\mathbf{d}}^Q) = X_{\sigma_i(\mathbf{d})}^{\sigma_i Q}.$$

2. AFFINE CLUSTER ALGEBRAS AND CLUSTER CATEGORIES

2.1. Cluster algebras of affine types. Fomin and Zelevinsky proved in [FZ03] that if Q is a quiver (or more generally a valued graph) of a given Dynkin type, then every acyclic quiver mutation-equivalent to Q is also a Dynkin quiver of the same Dynkin type. In this subsection, we generalize this result to the affine case.

Definition 2.1. A quiver Q is said to be of *affine type* $\tilde{A}_{r,s}$ if it is isomorphic to a quiver with underlying diagram of affine type \tilde{A}_{r+s-1} with r arrows going clockwise and s arrows going anticlockwise for some integers $r, s > 0$. It is said of *affine type* \tilde{D}_n (resp. \tilde{E}_n) if the underlying diagram is of type D_n (resp. E_n) for some integer $n \geq 4$ (resp. $n = 6, 7, 8$).

Remark 2.2. Note in particular that a quiver is of affine type $\tilde{A}_{r,s}$ if and only if it is of affine type $\tilde{A}_{s,r}$.

The following proposition proves that there is a finite-affine-wild classification of acyclic cluster algebras. It is actually a particular case of [CK06, Corollary 4].

Proposition 2.3. *Let Q be an acyclic quiver of a given affine type \tilde{X} . Then all the acyclic quivers mutation-equivalent to Q are of affine type \tilde{X} . Moreover, if Q' is a quiver of affine type \tilde{X} , then Q' is mutation equivalent to Q .*

Proof. If Q' is an acyclic quiver mutation-equivalent to Q , then Q' is the quiver of the opposite endomorphism ring of a cluster tilting object T in \mathcal{C}_Q . It thus follows from [KR08] that the cluster categories \mathcal{C}_Q and $\mathcal{C}_{Q'}$ are triangle equivalent. If we denote by $\mathbf{n} = (n_1, \dots, n_s)$ the tubular type of Q , then the AR-quiver of the cluster category of \mathcal{C}_Q contains s tubes of respective ranks n_1, \dots, n_s . Assume now that Q' is not of affine type. Then either Q' is Dynkin but then $\mathcal{C}_{Q'}$ has only finitely many non-isomorphic objects and thus is not equivalent to \mathcal{C}_Q , or Q' is wild and the AR-quiver of \mathcal{C}_Q does not contain tubes. It follows that Q' is also a quiver of affine type and moreover it has the same tubular type \mathbf{n} . Now, it is known that two affine quivers have the same tubular type if and only if they have the same affine type. This proves the first assertion.

For the second assertion, assume first that Q is a quiver of affine type $\tilde{A}_{r,s}$. If Q' is another quiver of type $\tilde{A}_{r,s}$, then it is known (see [ASS05] for example) that Q' is reflection-equivalent to Q and in particular, it is mutation-equivalent to Q . Now assume that Q and Q' are of affine type \tilde{D} or \tilde{E} . Then Q and Q' are two different orientations of a tree but it is known (see also [ASS05]) that any different orientations of a tree are reflection-equivalent. It follows that Q and Q' are mutation-equivalent. \square

Proposition 2.3 allows to speak of cluster algebras of affine type and to define the *affine type of such a cluster algebra*.

Definition 2.4. A cluster algebra \mathcal{A} is said to be of *affine type* \tilde{X} where $\tilde{X} = \tilde{A}_{r,s}, \tilde{D}_r$, or \tilde{E}_n for some non-negative integers r, s or some $n = 6, 7, 8$ if it contains a seed (\mathbf{x}, Q) where Q is an acyclic quiver of affine type \tilde{X} .

For simplicity, we say that \mathcal{A} is of affine type \tilde{A} if it is of affine type $\tilde{A}_{r,s}$ for some non-negative integers r, s .

Remark 2.5. Note that we exclude the quivers Q of cyclic types $\tilde{A}_{r,0}$ for some integer $r > 0$. Indeed, it is known that in this case, the cluster algebra is of Dynkin type \mathbb{D} .

2.2. Multiplications in affine cluster algebras. Following the ideas of [CK08] for cluster algebras of finite type, we will use cluster multiplication theorems in order to prove that $\mathcal{B}'(Q)$ generates the \mathbb{Z} -module $\mathcal{A}(Q)$. These theorems are fairly important for the study of cluster algebras because they provide a certain structure of Hall algebra for cluster categories. For different multiplication theorems, the reader can for example refer to [CK08, CK06, Dup09, Hub06, XX07, Xu10].

The following multiplication formula will be referred to as the *almost split multiplication formula*:

Proposition 2.6 ([CC06]). *Let Q be an acyclic quiver and M be a non-projective kQ -module. Then*

$$X_M X_{\tau M} = X_B + 1$$

where B is the central term of the almost split sequence

$$0 \longrightarrow \tau M \longrightarrow B \longrightarrow M \longrightarrow 0.$$

Another important formula is the *one-dimensional multiplication formula*:

Theorem 2.7 ([CK06]). *Let Q be an acyclic quiver and M, N be any two objects in \mathcal{C}_Q such that $\dim \text{Ext}_{\mathcal{C}_Q}^1(M, N) = 1$. Then*

$$X_M X_N = X_B + X_{B'}$$

where B and B' are the unique objects such that there exists non-split triangles

$$M \longrightarrow B \longrightarrow N \longrightarrow M[1] \text{ and } N \longrightarrow B' \longrightarrow M \longrightarrow N[1].$$

The following theorem is a consequence of [XX07] or [Xu10] (see also [Dup09, Theorem 9.2]).

Theorem 2.8. *Let Q be an acyclic quiver and M, N be two indecomposable kQ -modules. Then $X_M X_N$ can be written as a \mathbb{Q} -linear combination of X_Y where Y is either the middle term of short exact sequences of kQ -modules*

$$0 \longrightarrow M \longrightarrow Y \longrightarrow N \longrightarrow 0 \text{ or } 0 \longrightarrow N \longrightarrow Y \longrightarrow M \longrightarrow 0$$

or is isomorphic to $\ker f \oplus \text{coker } f[-1]$ for some morphism f in $\text{Hom}_{kQ}(M, \tau N)$ or in $\text{Hom}_{kQ}(N, \tau M)$.

Corollary 2.9. *Let Q be an acyclic quiver and M, N be any two objects in \mathcal{C}_Q . Then $X_M X_N$ can be written as a \mathbb{Q} -linear combination of X_Y where Y is the middle term of a non split triangle in $\text{Ext}_{\mathcal{C}_Q}^1(N, M)$ or in $\text{Ext}_{\mathcal{C}_Q}^1(M, N)$.*

Proof. We assume that $M, N \neq 0$. We prove it by induction on the number n of indecomposable direct summands of $M \oplus N$. If $n = 2$, then M and N are decomposable and the result is nothing but theorem 2.8. Assume that $n > 2$. Then $M \oplus N = M_1 \oplus M_2 \oplus N_1 \oplus N_2$ with M_1, N_1 indecomposable. Then

$$\begin{aligned} X_M X_N &= X_{M \oplus N} \\ &= X_{M_1 \oplus M_2 \oplus N_1 \oplus N_2} \\ &= X_{M_2 \oplus N_2} (X_{M_1} X_{N_1}) \\ &= X_{M_2 \oplus N_2} \sum_Y n_Y X_Y \end{aligned}$$

where the n_Y are rational numbers and Y runs over middle terms of non split triangles

$$M_1 \longrightarrow Y \longrightarrow N_1 \longrightarrow M_1[1] \text{ or } N_1 \longrightarrow Y \longrightarrow M_1 \longrightarrow N_1[1].$$

But if $M_1 \xrightarrow{i} Y \xrightarrow{p} N_1 \xrightarrow{\epsilon} M_1[1]$ is a non-split triangle, there is a non-split triangle

$$M_1 \oplus M_2 \xrightarrow{\begin{bmatrix} i & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} Y \oplus M_2 \oplus N_2 \xrightarrow{\begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} N_1 \oplus N_2 \xrightarrow{\begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}} M_1[1] \oplus M_2[1].$$

It follows that

$$X_M X_N = \sum_Y n_Y X_{Y \oplus M_2 \oplus N_2}$$

where n_Y are rational numbers and $Y \oplus M_2 \oplus N_2$ runs over middle terms of non split triangles

$$M \longrightarrow Y \oplus M_2 \oplus N_2 \longrightarrow N \longrightarrow M[1] \text{ or } N \longrightarrow Y \oplus M_2 \oplus N_2 \longrightarrow M \longrightarrow N[1].$$

□

Now, we get interested in middle terms occurring in these non-split triangles. For any two kQ -modules M, N , we write $\text{Ext}_{\mathcal{C}_Q}^1(N, M)_Z$ the set of triangles in $\text{Ext}_{\mathcal{C}_Q}^1(N, M)$ with middle term isomorphic to Z . For any object M in \mathcal{C}_Q , we write $[M, M]^1 = \dim \text{Ext}_{\mathcal{C}_Q}^1(M, M)$. We denote by $K_0^{\text{split}}(\mathcal{C}_Q)$ the split Grothendieck group, that is, the free abelian group over the isoclasses of \mathcal{C}_Q modulo the split triangles.

Lemma 2.10. *Let Q be an acyclic quiver, and M, N be two objects $\text{Ext}_{\mathcal{C}_Q}^1(N, M) \neq 0$. Fix $Z \neq M \oplus N$ an object such that $\text{Ext}_{\mathcal{C}_Q}^1(N, M)_Z \neq \emptyset$. Then*

$$[Z, Z]^1 < [M \oplus N, M \oplus N]^1$$

Proof. Fix a non-split triangle

$$M \longrightarrow Z \longrightarrow N \xrightarrow{\epsilon} M[1].$$

$[-, -]^1$ induces a bilinear form on $K_0^{\text{split}}(\mathcal{C}_Q)$. For any object R , the contravariant functor $\text{Hom}_{\mathcal{C}_Q}(-, R[1])$ applied to the triangle $M \longrightarrow Z \longrightarrow N \xrightarrow{\epsilon} M[1]$ gives rise to the exact sequence

$$0 \longrightarrow K_R \longrightarrow \text{Hom}_{\mathcal{C}_Q}(N, R[1]) \longrightarrow \text{Hom}_{\mathcal{C}_Q}(Z, R[1]) \longrightarrow \text{Hom}_{\mathcal{C}_Q}(M, R[1]) \longrightarrow C_R \longrightarrow 0$$

where K_R is the kernel of the map $\text{Hom}_{\mathcal{C}_Q}(N, R[1]) \longrightarrow \text{Hom}_{\mathcal{C}_Q}(Z, R[1])$ and C_R is the cokernel.

In particular, for any object R , we have

$$\dim K_R + \dim C_R + \dim \text{Hom}_{\mathcal{C}_Q}(Z, R[1]) = \dim \text{Hom}_{\mathcal{C}_Q}(N, R[1]) + \dim \text{Hom}_{\mathcal{C}_Q}(M, R[1]),$$

thus

$$\dim K_R + \dim C_R + [Z, R]^1 = [N, R]^1 + [M, R]^1.$$

We then get

$$[Z, N]^1 \leq [N, N]^1 + [M, N]^1 \text{ and } [Z, Z]^1 \leq [N, Z]^1 + [M, Z]^1.$$

Moreover, ϵ is a non-zero element in the kernel K_M of $\text{Hom}_{\mathcal{C}_Q}(N, M[1]) \longrightarrow \text{Hom}_{\mathcal{C}_Q}(Z, M[1])$, and thus $\dim K_M > 0$. Finally,

$$[Z, M]^1 < [N, M]^1 + [M, M]^1.$$

It follows that

$$\begin{aligned}
 [M + N, M + N]^1 &= [M, M]^1 + [N, N]^1 + [M, N]^1 + [N, M]^1 \\
 &> [Z, N]^1 + [Z, M]^1 \\
 &= [N, Z]^1 + [M, Z]^1 \\
 &\geq [Z, Z]^1
 \end{aligned}$$

which proves the lemma. \square

This implies the following corollary which will be essential to us:

Corollary 2.11. *Let Q be an affine quiver and M, N be two objects such that $\text{Ext}_{\mathcal{C}_Q}^1(N, M) \neq 0$. Then, $X_M X_N$ is a (finite) \mathbb{Q} -linear combination of X_Y where Y runs over objects in \mathcal{C}_Q such that $[Y, Y]^1 < [M \oplus N, M \oplus N]^1$.*

Proof. As τ is an auto-equivalence of \mathcal{C}_Q and \mathcal{C}_Q has infinitely many indecomposable objects in its transjective component, we can assume that M and N are modules. Then, every Y occurring in the expansion of corollary 2.9 is the middle term of a non-split triangle in the cluster category. Lemma 2.10 gives the result. \square

The following is an easy but useful lemma:

Lemma 2.12. *Let Q be an acyclic quiver and M, N be two objects in \mathcal{C}_Q . Then for any triangle*

$$M \longrightarrow E \longrightarrow N \longrightarrow M[1]$$

we have $\dim H^0(E) \leq \dim H^0(M \oplus N)$.

Proof. Applying H^0 to the triangle $M \xrightarrow{\alpha} E \xrightarrow{\beta} N \longrightarrow M[1]$, we get a long exact sequence

$$(2) \quad \cdots \longrightarrow H^0(M) \xrightarrow{H^0(\alpha)} H^0(E) \xrightarrow{H^0(\beta)} H^0(N) \longrightarrow \cdots$$

We denote by

$$\begin{aligned}
 K_\alpha &= \ker H^0(\alpha), \quad C_\alpha = \text{coker } H^0(\alpha), \\
 K_\beta &= \ker H^0(\beta), \quad C_\beta = \text{coker } H^0(\beta).
 \end{aligned}$$

We have two exact sequences

$$\begin{aligned}
 0 \longrightarrow K_\alpha \longrightarrow H^0(M) \xrightarrow{H^0(\alpha)} H^0(E) \longrightarrow C_\alpha \longrightarrow 0, \\
 0 \longrightarrow K_\beta \longrightarrow H^0(E) \xrightarrow{H^0(\beta)} H^0(N) \longrightarrow C_\beta \longrightarrow 0.
 \end{aligned}$$

In the Grothendieck group $K_0(kQ)$, we thus get the equalities

$$\begin{aligned}
 H^0(E) &= H^0(M) + C_\alpha - K_\alpha \\
 H^0(E) &= H^0(N) + K_\beta - C_\beta
 \end{aligned}$$

and thus

$$2H^0(E) = H^0(M) + H^0(N) + (C_\alpha - C_\beta) + (K_\beta - K_\alpha)$$

As the sequence (2) is exact, we have $K_\beta = \text{Im}(H^0(\alpha))$ and thus $H^0(E) - K_\beta = C_\alpha$. It follows that

$$H^0(E) = H^0(M) + H^0(N) - (K_\alpha + C_\beta)$$

and thus identifying $K_0(kQ)$ and \mathbb{Z}^{Q_0} with the dimension vector, we get

$$\dim H^0(E) \leq \dim H^0(M \oplus N).$$

\square

Lemma 2.13. *Let Q be an affine quiver, \mathcal{T} be a tube of the AR-quiver of Q and M, N be two indecomposable regular modules in \mathcal{T} . Then for any triangle*

$$M \longrightarrow E \longrightarrow N \longrightarrow M[1]$$

E is a regular module in \mathcal{T} .

Proof. Fix a triangle $M \xrightarrow{\alpha} E \xrightarrow{\beta} N \xrightarrow{\phi} M[1]$. With notations of lemma 2.12, we have

$$H^0(E) = H^0(M) + H^0(N) - (K_\alpha + C_\beta).$$

Thus, $\partial_{H^0(E)} = \partial_{H^0(M)} + \partial_{H^0(N)} - \partial_{K_\alpha} - \partial_{C_\beta} = -\partial_{K_\alpha} - \partial_{C_\beta}$. The exact sequences

$$0 \longrightarrow K_\alpha \longrightarrow H^0(M) \xrightarrow{H^0(\alpha)} H^0(E) \longrightarrow C_\alpha \longrightarrow 0,$$

$$0 \longrightarrow K_\beta \longrightarrow H^0(E) \xrightarrow{H^0(\beta)} H^0(N) \longrightarrow C_\beta \longrightarrow 0.$$

yield

$$\begin{cases} \partial_{H^0(E)} &= \partial_{C_\alpha} - \partial_{K_\alpha} \\ \partial_{H^0(E)} &= \partial_{K_\beta} - \partial_{C_\beta} \end{cases}$$

so that

$$\begin{cases} \partial_{C_\beta} &= -\partial_{C_\alpha} \\ \partial_{K_\beta} &= -\partial_{K_\alpha}. \end{cases}$$

Set $f = H^0(\phi) : N \longrightarrow \tau M$, it is a morphism between regular modules. It follows that $\partial_{\ker f} = 0 = \partial_{\text{coker } f}$. But from the long exact sequence 2, we get $\text{Im}(H^0(\beta)) = \ker f$ so that the exact sequence

$$0 \longrightarrow \text{Im}(H^0(\beta)) \longrightarrow H^0(N) \simeq N \longrightarrow C_\beta \longrightarrow 0$$

gives $\partial_{C_\beta} = 0$ and thus $\partial_{H^0(E)} = \partial_{K_\beta}$. On the other hand, $K_\alpha = \ker H^0(\alpha) = \text{Im}(H^0(f[-1])) \simeq \text{Im}(H^0(\tau f))$ but τf is a morphism between regular modules so $\text{Im}(\tau f)$ is a regular module and thus $\partial_{K_\alpha} = 0$. As $\partial_{K_\alpha} = -\partial_{K_\beta} = -\partial_{H^0(E)}$, it follows that $\partial_{H^0(E)} = 0$.

Write $E = P_E[1] \oplus P \oplus R \oplus I$ where P_E is a projective module, P is a preprojective module, R is a regular module and I is a preinjective module. Assume that I is non-zero, thus by defect, P is also non-zero. Denote by p the rank of \mathcal{T} , there exists some $k \in \mathbb{Z}$ such that $\tau^{-k}I = 0$. Thus $I[-k]$ is either the shift of a projective module, or a preprojective module. Note that $P[-k]$ is a non-zero module. Consider the triangle associated to the morphism $\phi[-k]$, it is given by $M \xrightarrow{\alpha[-k]} E[-k] \xrightarrow{\beta[-k]} N \xrightarrow{\phi[-k]} M[1]$. It follows from the above discussion that $\partial_{H^0(E[-k])} = 0$ but

$$E[-k] = P_E[1-k] \oplus P[-k] \oplus R[-k] \oplus I[-k]$$

so that $P_E[1-k] \oplus P[-k] \oplus I[-k] = 0$ and thus $P_E = P = I = 0$, which is a contradiction. It follows that $E = R$ is a regular module. Moreover, as there are neither morphisms nor extensions between different tubes, it follows that $E \in \text{add } \mathcal{T}$. \square

Proposition 2.14. (1) *Let Q be an affine quiver and M be an object in \mathcal{C}_Q . Then X_M can be written as a \mathbb{Q} -linear combination of X_Y where Y is an object such that $\dim H^0(Y) \leq \dim H^0(M)$ and $\text{Ext}_{\mathcal{C}_Q}^1(Y_i, Y_j) = 0$ for any two distinct direct summands of Y .*

(2) *If moreover M is taken in $\text{add } \mathcal{T}$ for some tube \mathcal{T} of the AR-quiver of Q , then the Y can be taken also in $\text{add } \mathcal{T}$.*

Proof. We prove the it assertion by induction on the dimension of the self-extension. If M is rigid, the result holds. Assume now that M is not rigid. If M is indecomposable, the result holds so we can assume that $M = M_1 \oplus M_2$ with $[M_1, M_2]^1 \neq 0$. Corollary 2.11 implies that $X_M = X_{M_1} X_{M_2}$ is a \mathbb{Q} -linear combination of X_Y with $[Y, Y]^1 < [M_1 \oplus M_2, M_1 \oplus M_2] = [M, M]^1$. Thus, by induction each X_Y is a \mathbb{Q} -linear combination of X_Z where Z is such that $\text{Ext}_{\mathcal{C}_Q}^1(Z_i, Z_j) = 0$ for any two distinct direct summands of Z . The fact that $\dim H^0(Y) \leq \dim H^0(M)$ follows directly from lemma 2.12. This proves the first point. The second point follows from lemma 2.13. \square

3. GENERIC VARIABLES

3.1. Generic generalized variables for acyclic quivers. In this subsection, we introduce the notion of generic variables, which will be the central objects in this article. They will turn out to be good candidates to construct a \mathbb{Z} -basis in a cluster algebra of affine type. As in the work of [GLS05], these elements will be generic values for some particular constructible functions. The following construction holds for any acyclic quiver.

3.1.1. Existence of generic variables. Let Q be an acyclic quiver, we denote by Q_0 the set of vertices, Q_1 the set of arrows. For any $\alpha : i \rightarrow j \in Q_1$, we denote by $s(\alpha) = i$ the source of α and $t(\alpha) = j$ the tail of α . For any dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $\text{rep}(Q, \mathbf{d})$ the *representation variety of dimension drepresentation variety*, it is isomorphic to

$$\prod_{\alpha \in Q_1} k^{d_{s(\alpha)}} \times k^{d_{t(\alpha)}},$$

in particular it is an irreducible affine variety.

For the existence of subset of $\text{rep}(Q, \mathbf{d})$ giving a generic value to the Caldero-Chapoton map, we prove the existence of generic values for Euler characteristics of submodule grassmannians.

The following lemma will be essential:

Lemma 3.1. *Fix Q an acyclic quiver, \mathbf{d} a dimension vector. For every dimension vector \mathbf{e} , there exists a dense open subset $U_{\mathbf{d}, \mathbf{e}} \subset \text{rep}(Q, \mathbf{d})$ such that the map*

$$M \mapsto \chi(\text{Gr}_{\mathbf{e}}(M))$$

is constant over $U_{\mathbf{d}, \mathbf{e}}$. Moreover, if $U'_{\mathbf{d}, \mathbf{e}}$ is another such dense open subset, the value is the same on $U_{\mathbf{d}, \mathbf{e}}$ and $U'_{\mathbf{d}, \mathbf{e}}$.

Proof. We consider the algebraic variety

$$V_{\mathbf{e}} = \{(M, N) \in \text{rep}(Q, \mathbf{d}) \times \text{rep}(Q, \mathbf{e}) : N \text{ is a submodule of } M\}$$

We denote by $f_{\mathbf{e}} : V_{\mathbf{e}} \rightarrow \text{rep}(Q, \mathbf{d})$ the first projection. For any representation M , the grassmannian $\text{Gr}_{\mathbf{e}}(M)$ is the fiber of M under the morphism $f_{\mathbf{e}}$. If $f_{\mathbf{e}}$ is a dominant morphism, then according to [Ver76, Corollary 5.1], there exists a non-empty subset $U_{\mathbf{d}, \mathbf{e}} \subset \text{rep}(Q, \mathbf{d})$, open for the Zariski topology (and thus dense by irreducibility) such that the fibers of $f_{\mathbf{e}}$ are isomorphic on $U_{\mathbf{d}, \mathbf{e}}$. In particular, they have the same Euler characteristic. Now if $f_{\mathbf{e}}$ is not dominant, then $\text{codim}(\text{Im } f_{\mathbf{e}}) > 0$ and thus the Euler characteristic of $\text{Gr}_{\mathbf{e}}(M)$ vanishes on an open subset of $\text{rep}(Q, \mathbf{d})$. Assume now that $U'_{\mathbf{d}, \mathbf{e}}$ is another such subset. Then by irreducibility $U_{\mathbf{d}, \mathbf{e}} \cap U'_{\mathbf{d}, \mathbf{e}} \neq \emptyset$ and thus the values coincide. \square

Notation 3.2. Fix $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0}$, we will use the following notations:

$$\mathbf{e} \leq \mathbf{d} \Leftrightarrow e_i \leq d_i \text{ for all } i \in Q_0$$

$$\mathbf{e} \preceq \mathbf{d} \Leftrightarrow e_i \leq d_i \text{ for all } i \in Q_0 \text{ and } \mathbf{e} \neq \mathbf{d}$$

If $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we write

$$\begin{aligned} [\mathbf{d}]_+ &= (\max(d_i, 0))_{i \in Q_0} \\ [\mathbf{d}]_- &= (\min(d_i, 0))_{i \in Q_0} \end{aligned}$$

Note that if M is a rigid object in \mathcal{C}_Q , we have

$$[\dim M]_+ = \dim H^0(M).$$

Indeed, if M is a module the result is clear. Now assume that $P_i[1]$ is a direct summand of M , then

$$\dim H^0(M)(i) = \dim \operatorname{Hom}_{kQ}(P_i, H^0(M)) \leq \dim \operatorname{Ext}_{\mathcal{C}_Q}^1(H^0(M), P_i) = 0.$$

Corollary 3.3. *Fix Q an acyclic quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then there exists a dense open subset $U_{\mathbf{d}} \subset \operatorname{rep}(Q, \mathbf{d})$ such that the Caldero-Chapoton map is constant over $U_{\mathbf{d}}$. Moreover, if $U'_{\mathbf{d}}$ is another such open subset, the values of the Caldero-Chapoton map are the same on $U_{\mathbf{d}}$ and $U'_{\mathbf{d}}$.*

Proof. Consider

$$U_{\mathbf{d}} = \bigcap_{\mathbf{e} \leq \mathbf{d}} U_{\mathbf{d}, \mathbf{e}}.$$

It is a finite intersection of dense open subsets, it is thus open and dense in $\operatorname{rep}(Q, \mathbf{d})$. If $\mathbf{e} \leq \mathbf{d}$, we denote by $n_{\mathbf{e}}$ the value of $\chi(\operatorname{Gr}_{\mathbf{e}}(-))$ on $U_{\mathbf{d}, \mathbf{e}}$. For any $M \in U_{\mathbf{d}}$, we thus have

$$\begin{aligned} X_M &= \sum_{\mathbf{e} \leq \mathbf{d}} \chi(\operatorname{Gr}_{\mathbf{e}}(M)) \prod_i u_i^{-\langle \mathbf{e}, \alpha_i \rangle - \langle \alpha_i, \mathbf{d} - \mathbf{e} \rangle} \\ &= \sum_{\mathbf{e} \leq \mathbf{d}} n_{\mathbf{e}} \prod_i u_i^{-\langle \mathbf{e}, \alpha_i \rangle - \langle \alpha_i, \mathbf{d} - \mathbf{e} \rangle} \end{aligned}$$

it is thus constant over $U_{\mathbf{d}}$.

Assume now that $U'_{\mathbf{d}}$ is another such subset, then $U_{\mathbf{d}} \cap U'_{\mathbf{d}} \neq \emptyset$ and the values coincide on both subsets. \square

Note that for every \mathbf{d} , the set $U_{\mathbf{d}}$ is a $G_{\mathbf{d}}$ -invariant subset of $\operatorname{rep}(Q, \mathbf{d})$.

Definition 3.4. Fix $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $X_{\mathbf{d}}$ the value of the Caldero-Chapoton map on $U_{\mathbf{d}}$. Fix now $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we set

$$\begin{aligned} X_{\mathbf{d}} &= X_{[\mathbf{d}]_+} \prod_{d_i < 0} u_i^{-d_i} \\ &= X_{[\mathbf{d}]_+} \prod_{d_i < 0} X_{P_i[1]}^{-d_i} \\ &= X_{[\mathbf{d}]_+} \prod_{d_i < 0} X_{P_i[1] \oplus (-d_i)} \end{aligned}$$

and $X_{\mathbf{d}}$ is called the *generic variable of dimension \mathbf{d}* .

3.1.2. Generic variables and canonical decomposition. Fix Q an acyclic quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector. According to [Kac80, Kac82], there exists a dense open subset $\mathfrak{M}_{\mathbf{d}} \subset \operatorname{rep}(Q, \mathbf{d})$ and a family of Schur root $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in \mathbb{N}^{Q_0} such that every representation $M \in \mathfrak{M}_{\mathbf{d}}$ decomposes into

$$M = \bigoplus_{i=1}^n M_i$$

with M_i an indecomposable Schur representation of dimension \mathbf{e}_i . In particular $\mathbf{d} = \sum_{i=1}^n \mathbf{e}_i$ and the family $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is uniquely determined. This decomposition of \mathbf{d} is called the *canonical decomposition of \mathbf{d}* and is denoted by

$$\mathbf{d} = \bigoplus_{i=1}^n \mathbf{e}_i.$$

We have the following characterization:

Proposition 3.5 ([Kac82]). *Fix Q an acyclic quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then $\mathbf{d} = \bigoplus_{i=1}^n \mathbf{e}_i$ if and only if there exist Schur representations $M_i \in \text{rep}(Q, \mathbf{e}_i)$ for every $i = 1, \dots, n$ such that $\text{Ext}_{kQ}^1(M_i, M_j) = 0$ if $i \neq j$.*

Following Schofield, we set the definition:

Definition 3.6. For any $\mathbf{d}, \mathbf{d}' \in \mathbb{N}^{Q_0}$, we say that $\text{Ext}_{kQ}^1(\mathbf{d}, \mathbf{d}')$ *vanishes generally* if there exists $M \in \text{rep}(Q, \mathbf{d})$, $M' \in \text{rep}(Q, \mathbf{d}')$ such that

$$\text{Ext}_{kQ}^1(M, M') = 0$$

and if it is the case, we denote it by $\text{Ext}_{kQ}^1(\mathbf{d}, \mathbf{d}') = 0$.

For any dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, one has $U_{\mathbf{d}} \cap \mathfrak{M}_{\mathbf{d}} \neq \emptyset$, thus $X_{\mathbf{d}}$ is equal to a X_M for some module $M \in \mathfrak{M}_{\mathbf{d}}$. One can thus use the canonical decomposition to compute generic variables. More precisely, a helpful result is the following:

Proposition 3.7. *Fix Q an acyclic quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector with canonical decomposition $\mathbf{d} = \mathbf{e}_1 \oplus \dots \oplus \mathbf{e}_n$. Then*

$$X_{\mathbf{d}} = \prod_{i=1}^n X_{\mathbf{e}_i}.$$

Proof. Consider the injective morphism :

$$\phi : \begin{cases} \text{rep}(Q, \mathbf{e}_1) \times \dots \times \text{rep}(Q, \mathbf{e}_n) & \longrightarrow & \text{rep}(Q, \mathbf{d}) \\ (M_1, \dots, M_n) & \mapsto & \bigoplus_{i=1}^n M_i \end{cases}$$

By proposition 3.5, as $\mathbf{d} = \mathbf{e}_1 \oplus \dots \oplus \mathbf{e}_n$ is the canonical decomposition of \mathbf{d} , one has $\text{Ext}_{kQ}^1(\mathbf{e}_i, \mathbf{e}_j) = 0$ for every $i \neq j$. It thus follows from [CBS02] that ϕ is a dominant morphism.

We set

$$\mathcal{U} = (\mathfrak{M}_{\mathbf{e}_1} \cap U_{\mathbf{e}_1}) \times \dots \times (\mathfrak{M}_{\mathbf{e}_n} \cap U_{\mathbf{e}_n}),$$

this is a dense open subset in $\text{rep}(Q, \mathbf{e}_1) \times \dots \times \text{rep}(Q, \mathbf{e}_n)$ and thus $\phi(\mathcal{U})$ is a dense open subset in $\text{rep}(Q, \mathbf{d})$. In particular $\phi(\mathcal{U}) \cap U_{\mathbf{d}} \neq \emptyset$ and we can choose some $M \in \phi(\mathcal{U}) \cap U_{\mathbf{d}}$. We thus have $X_{\mathbf{d}} = X_M$ and M decomposes into a direct sum

$$M = \bigoplus_{i=1}^n M_i$$

with $M_i \in \mathfrak{M}_{\mathbf{e}_i} \cap U_{\mathbf{e}_i}$. It follows that

$$\begin{aligned} X_{\mathbf{d}} &= X_M \\ &= X_{\bigoplus_{i=1}^n M_i} \\ &= \prod_{i=1}^n X_{M_i} \\ &= \prod_{i=1}^n X_{\mathbf{e}_i}. \end{aligned}$$

□

The following lemma gives a multiplicative property for generic variables. For this, we extend the notion of general vanishing of the Ext spaces to the cluster category.

Definition 3.8. Let \mathbf{d}, \mathbf{d}' be two elements in \mathbb{Z}^{Q_0} , we say that $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}')$ *vanishes generally* if there exists $M \in \text{rep}(Q, [\mathbf{d}]_+)$, $M' \in \text{rep}(Q, [\mathbf{d}']_+)$ such that

$$\text{Ext}_{\mathcal{C}_Q}^1 \left(M \oplus \bigoplus_{d_i < 0} P_i[1]^{\oplus(-d_i)}, M' \oplus \bigoplus_{d'_i < 0} P_i[1]^{\oplus(-d'_i)} \right) = 0$$

and we denote this by $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}') = 0$.

Note in particular that for $\mathbf{d}, \mathbf{d}' \in \mathbb{N}^{Q_0}$, if $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}') = 0$ then $\text{Ext}_{kQ}^1(\mathbf{d}, \mathbf{d}') = 0$ and $\text{Ext}_{kQ}^1(\mathbf{d}', \mathbf{d}) = 0$.

Lemma 3.9. *Let Q be an acyclic quiver and $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^{Q_0}$ such that $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}') = 0$. Then*

$$X_{\mathbf{d}} X_{\mathbf{d}'} = X_{\mathbf{d} + \mathbf{d}'}.$$

Proof. Assume that there is some $i \in Q_0$ such that $d_i < 0$ and $d'_i > 0$, then for every $M' \in \text{rep}(Q, \mathbf{d}')$, $0 < d'_i = \dim \text{Hom}_{kQ}(P_i, M') = \dim \text{Ext}_{\mathcal{C}_Q}^1(M', P_i[1])$ and thus $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}')$ does not vanish generally. Thus, we can assume that d_i and d'_i are of the same sign for every $i \in Q_0$, in particular, $[\mathbf{d} + \mathbf{d}']_+ = [\mathbf{d}]_+ + [\mathbf{d}']_+$. As $\text{Ext}_{\mathcal{C}_Q}^1(\mathbf{d}, \mathbf{d}') = 0$, we have $\text{Ext}_{kQ}^1(\mathbf{d}, \mathbf{d}') = 0$ and $\text{Ext}_{kQ}^1(\mathbf{d}', \mathbf{d}) = 0$, it follows from [CBS02] that the morphism

$$\phi : \begin{cases} \text{rep}(Q, [\mathbf{d}]_+) \times \text{rep}(Q, [\mathbf{d}']_+) & \longrightarrow & \text{rep}(Q, [\mathbf{d} + \mathbf{d}']_+) \\ (U, V) & \mapsto & U \oplus V \end{cases}$$

is dominant. As $U_{[\mathbf{d}]}$ (resp $U_{[\mathbf{d}']_+}$) is open in $\text{rep}(Q, [\mathbf{d}]_+)$ (resp. in $\text{rep}(Q, [\mathbf{d}']_+)$), it follows that $U_{[\mathbf{d}]} \oplus U_{[\mathbf{d}']_+}$ is open and dense in $\text{rep}(Q, [\mathbf{d} + \mathbf{d}']_+)$. Thus, $X_{[\mathbf{d} + \mathbf{d}']_+} = X_{[\mathbf{d}]_+} X_{[\mathbf{d}']_+}$ and

$$\begin{aligned} X_{\mathbf{d}} X_{\mathbf{d}'} &= X_{[\mathbf{d}]_+} \left(\prod_{d_i < 0} X_{P_i[1]^{\oplus(-d_i)}} \right) X_{[\mathbf{d}']_+} \left(\prod_{d'_i < 0} X_{P_i[1]^{\oplus(-d'_i)}} \right) \\ &= X_{[\mathbf{d} + \mathbf{d}']_+} \prod_{d_i + d'_i < 0} X_{P_i[1]^{\oplus(-(d_i + d'_i))}} \\ &= X_{\mathbf{d} + \mathbf{d}'} \end{aligned}$$

□

The following proposition proves that generic variables are indeed generalizing cluster monomials.

Lemma 3.10. *Let Q be an acyclic quiver, fix x a cluster monomial in $\mathcal{A}(Q)$ with denominator vector $\delta(x)$. Then*

$$x = X_{\delta(x)}.$$

Equivalently, if M is a rigid object, we have

$$X_M = X_{\dim M}.$$

Proof. According to theorem 1.3, if x is a cluster monomial, it can be written X_M for some rigid object M and theorem 1.2 implies that $\delta(X_M) = \dim M$. Moreover, as M is rigid, we have $[\dim M]_+ = \dim H^0(M)$ and $[\dim M]_- = \dim P_M[1]$. $H^0(M)$ being a rigid module in $\text{rep}(Q, [\dim M]_+)$, its orbit $\mathcal{O}_{H^0(M)}$ is dense in

$\text{rep}(Q, [\mathbf{dim} M]_+)$ and thus $\mathcal{O}_{H^0(M)} \cap U_{[\mathbf{dim} M]_+} \neq \emptyset$. This implies that $X_{H^0(M)} = X_{[\mathbf{dim} M]_+}$. Now, we always have $X_{P_M[1]} = X_{[\mathbf{dim} M]_-}$. It follows that

$$\begin{aligned} x &= X_M \\ &= X_{H^0(M) \oplus P_M[1]} \\ &= X_{H^0(M)} X_{P_M[1]} \\ &= X_{[\mathbf{dim} M]_+} X_{[\mathbf{dim} M]_-} \end{aligned}$$

but $\text{Ext}_{\mathcal{C}_Q}^1([\mathbf{dim} M]_+, [\mathbf{dim} M]_-)$ vanishes generally because M is a rigid module and so lemma 3.9 implies that

$$\begin{aligned} X_{[\mathbf{dim} M]_+} X_{[\mathbf{dim} M]_-} &= X_{[\mathbf{dim} M]_+ + [\mathbf{dim} M]_-} \\ &= X_{\mathbf{dim} M} \\ &= X_{\delta(x)} \end{aligned}$$

which proves the lemma. \square

Through the works of [CK08, SZ04, GLS08, CZ06], it became clear that the \mathbb{Z} -bases for cluster algebras should naturally contain the cluster monomials. Lemma 3.10 proves that they turn out to be generic variables. Moreover, the analogy with the works of [GLS08] in the context of preprojective algebras gives us the hope that the set of generic variables is a good candidate for being a \mathbb{Z} -basis in cluster algebras. We thus get interested in the set of all generic variables.

Definition 3.11. Let Q be an acyclic quiver, we denote by $\mathcal{B}'(Q)$ the set of all generic variables

$$\mathcal{B}'(Q) = \{X_{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{Q_0}\}.$$

3.2. Generic variables for affine quivers. In this subsection, we obtain an explicit description of the generic variables when Q is a quiver of affine type. We denote by δ its minimal imaginary root. We recall that the positive root system of Q can be written

$$\Phi_{\geq 0}(Q) = \Phi_{\geq 0}^{\text{re}}(Q) \sqcup \Phi_{\geq 0}^{\text{im}}(Q)$$

where $\Phi_{\geq 0}^{\text{re}}(Q)$ is the set of positive real roots and $\Phi_{\geq 0}^{\text{im}}(Q) = \mathbb{N}\delta$ is the set of imaginary roots. For details concerning representation theory of affine quivers, one can for example refer to [Rin84, DR76].

3.2.1. Generic variables associated to positive real Schur roots.

Lemma 3.12. Let Q be an affine quiver and \mathbf{d} be a positive real Schur root of Q . Then there exists a unique indecomposable rigid module M of dimension \mathbf{d} . Moreover, $X_{\mathbf{d}} = X_M$.

Proof. As \mathbf{d} is a real root, Kac's theorem ensures that there exists a unique indecomposable representation M of dimension \mathbf{d} . Moreover, as \mathbf{d} is a Schur root, this representation has a trivial endomorphism ring. Now $1 = \langle \mathbf{d}, \mathbf{d} \rangle = \langle M, M \rangle = \dim \text{End}_{kQ}(M) - \dim \text{Ext}_{kQ}^1(M, M)$ so $\dim \text{Ext}_{kQ}^1(M, M) = 0$. It follows from lemma 3.10 that $X_M = X_{\mathbf{dim} M} = X_{\mathbf{d}}$. \square

3.2.2. Generic variables associated to positive imaginary roots. We write $\mathbb{P}^1 = \mathbb{P}^1(k)$ the index set of tubes in the AR quiver of kQ and by \mathbb{P}_0^1 the index set of homogeneous tubes. For any $\lambda \in \mathbb{P}^1$, we will denote by \mathcal{T}_λ the tube indexed by λ . If $\lambda \in \mathbb{P}_0^1$, M_λ denotes the quasi-simple in \mathcal{T}_λ and $M_\lambda^{(n)}$ denotes the indecomposable module in \mathcal{T}_λ with quasi-socle M_λ and quasi-length n . In particular, $\mathbf{dim} M_\lambda^{(n)} = n\delta$ for any $n \geq 1$ and $\lambda \in \mathbb{P}_0^1$.

It is convenient to introduce the so-called *normalized Chebyshev polynomials of the second kind* (called *generalized Chebyshev polynomials of rank 2* in [Dup09]). For any $n \geq 0$, the n -th normalized Chebyshev polynomial of the second kind is the polynomial C_n defined by

$$C_n(t + t^{-1}) = \sum_{k=0}^n t^{n-2k}$$

One can refer to subsection 5.2 (see also [CZ06] and [Dup09]) for details concerning these polynomials.

The following lemma is proved in [CZ06], we recall it for completeness.

Lemma 3.13. *Fix Q an affine quiver, $n \geq 1$ and $\lambda \in \mathbb{P}_0^1$. Then*

$$X_{M_\lambda^{(n)}} = C_n(X_{M_\lambda})$$

Proof. For any $n \geq 1$ and any $\lambda \in \mathbb{P}_0^1$, we have an almost split sequence

$$0 \longrightarrow M_\lambda^{(n)} \longrightarrow M_\lambda^{(n+1)} \oplus M_\lambda^{(n-1)} \longrightarrow M_\lambda^{(n)} \longrightarrow 0.$$

The almost split multiplication formula implies thus that

$$X_{M_\lambda^{(n)}}^2 = X_{M_\lambda^{(n-1)}} X_{M_\lambda^{(n+1)}} + 1$$

and thus that $M_\lambda^{(n)} = C_n(M_\lambda)$ where C_n is the second kind normalized Chebyshev polynomial. \square

We now prove that for an affine quiver, the Caldero-Chapoton map does not depend on the considered homogeneous tube.

Lemma 3.14. *Let Q be an affine quiver of affine type. Then for any $\lambda, \mu \in \mathbb{P}_0^1$ and any $n \geq 1$, we have*

$$X_{M_\lambda^{(n)}} = X_{M_\mu^{(n)}}$$

Proof. By lemma 3.13, it is enough to prove it for $n = 1$. For this, we recall Crawley-Boevey's construction of regular representations of dimension δ by one-point extensions [CB92]. Let e be an extending vertex of Q , that is a vertex $e \in Q_0$ such that $\delta_e = 1$ and that the quiver obtained from Q by deleting the vertex e is of Dynkin type. We denote by $P = P_e$ the projective module associated to the vertex e and by $\mathbf{p} = \mathbf{dim} P$ its dimension vector. Then the defect of P is $\partial_P = -1$. Let L be the unique indecomposable representation of dimension vector $\delta + \mathbf{p}$. Then L has also defect -1 . Crawley-Boevey proved that $\text{Hom}_{kQ}(P, L) \simeq k^2$ and that for every morphism $0 \neq \lambda \in \text{Hom}_{kQ}(P, L)$, $\text{coker } \lambda$ is an indecomposable regular module of dimension vector δ . Moreover, $\text{coker } \lambda \simeq \text{coker } \lambda'$ if and only if $\lambda = x\lambda'$ for some $0 \neq x \in k$. Then $\lambda \mapsto \text{coker } \lambda$ induces a bijection from $\mathbb{P}\text{Hom}_{kQ}(P, L)$ to the set of all tubes of Q by sending $\lambda \in \mathbb{P}\text{Hom}_{kQ}(P, L)$ to an indecomposable regular module of dimension δ . We thus identify $\mathbb{P}\text{Hom}_{kQ}(P, L)$ and \mathbb{P}^1 and we will say that $\lambda \in \mathbb{P}_0^1$ to signify that $\text{coker } \lambda$ is a quasi-simple in the homogeneous tube \mathcal{T}_λ . Note that in this case $M_\lambda \simeq \text{coker } \lambda$.

Fix $0 \neq \lambda \in \text{Hom}_{kQ}(P, L)$, then

$$\begin{aligned} \text{Ext}_{\mathcal{C}_Q}^1(P, \text{coker } \lambda) &\simeq \text{Ext}_{kQ}^1(P, \text{coker } \lambda) \oplus \text{Ext}_{kQ}^1(\text{coker } \lambda, P) \\ &\simeq \text{Ext}_{kQ}^1(\text{coker } \lambda, P) \\ &\simeq \text{Hom}_{kQ}(P, \text{coker } \lambda) \\ &\simeq k \end{aligned}$$

because $P = P_e$ and e is an extending vertex.

It follows from Caldero-Keller's multiplication formula that

$$X_P X_{\text{coker } \lambda} = X_L + X_B$$

where $B \simeq \ker f \oplus \text{coker } f[-1]$ for some morphism $0 \neq f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$.

Fix $\lambda \in \mathbb{P}_0^1$. Assume that f is not surjective. Then $\text{Im } f$ is preprojective and we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow P \xrightarrow{f} \text{Im } f \longrightarrow 0$$

and thus $\partial_{\ker f} + \partial_{\text{Im } f} = \partial_P = -1$ but $\ker f$ and $\text{Im } f$ are preprojective so necessarily $\partial_{\ker f} = 0$ and $\ker f = 0$. It follows that $B \simeq \tau^{-1} \text{coker } f$. But

$$\partial_{\text{coker } f} = \partial_{(\tau \text{coker } \lambda)/P} = -\partial_P = 1$$

and $\tau \text{coker } \lambda \simeq \text{coker } \lambda$ is quasi-simple in an homogeneous tube. Thus, $\text{coker } f$ is preinjective and by defect, it is indecomposable. Moreover, $\mathbf{dim} \text{coker } f = \delta - \mathbf{p}$. Thus, for every $\lambda \in \mathbb{P}_0^1$ and every non-surjective map $0 \neq f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$, $\text{coker } f$ is the unique indecomposable representation V of dimension $\delta - \mathbf{p}$. If there exists some non-surjective $f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$, then f is injective, so $\mathbf{dim} P \not\leq \delta$ and thus every map $g \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$ is non-surjective for any $\lambda \in \mathbb{P}_0^1$. Thus,

$$X_P X_{\text{coker } \lambda} = X_L + X_V[-1]$$

for any $\lambda \in \mathbb{P}_0^1$. In particular, X_{M_λ} does not depend on the parameter $\lambda \in \mathbb{P}_0^1$.

Assume now that $0 \neq f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$ is surjective. Then $\ker f$ is a preprojective module of dimension $\mathbf{p} - \delta$. Thus, $\ker f$ has defect -1 , it is thus necessarily indecomposable. As there exists an unique indecomposable representation U of dimension $\mathbf{p} - \delta$, it follows that $\ker f \simeq U$ for every $\lambda \in \mathbb{P}_0^1$ and every surjective $f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$. Then, it follows from the above discussion that if one of the $f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$ is surjective for some $\lambda \in \mathbb{P}_0^1$, then every $f \in \text{Hom}_{kQ}(P, \tau \text{coker } \lambda)$ is surjective for any $\lambda \in \mathbb{P}_0^1$. In this case, we get

$$X_P X_{\text{coker } \lambda} = X_L + X_U$$

for every $\lambda \in \mathbb{P}_0^1$. In particular, X_{M_λ} does not depend on the parameter $\lambda \in \mathbb{P}_0^1$. \square

If E is a quasi-simple module in an exceptional tube \mathcal{T} , we denote by $E^{(n)}$ the indecomposable module with quasi-socle E and quasi-length n . If p is the rank of \mathcal{T} and $k \geq 1$ is an integer, we will simplify the notations by writing $M_E^{(k)} = E^{(kp)}$ and $M_E = E^{(p)}$. Note in particular that $\mathbf{dim} M_E^{(k)} = k\delta$.

Lemma 3.15. *Fix $n \geq 1$ a positive integer. Then*

$$X_{n\delta} = X_{M_\lambda}^n$$

for any $\lambda \in \mathbb{P}_0^1$.

Proof. We first prove it for $n = 1$. The canonical decomposition of δ is δ itself. It follows that $X_\delta = X_M$ for some indecomposable module M of dimension vector δ . Now, we know that the indecomposable modules of dimension vector δ are either the M_λ for $\lambda \in \mathbb{P}_0^1$, or the M_E for E quasi-simple in an exceptional tube. Fix now a quasi-simple E in an exceptional tube, we claim that $\text{Gr}_{\mathbf{dim } E}(M) = \emptyset$ for any

indecomposable module M of dimension vector δ not isomorphic to M_E . Indeed, fix $\lambda \in \mathbb{P}_0^1$ and $U \subset M_\lambda$ a submodule such that $\mathbf{dim} U = \mathbf{dim} E$. As M_λ is quasi-simple, U has to be preprojective and thus $\partial_U < 0$ but $\partial_U = \partial_E = 0$, which is a contradiction and thus $\mathrm{Gr}_{\mathbf{dim} E}(M_\lambda) = \emptyset$. Fix now F another quasi-simple and assume that $U \subset M_F$ is a submodule such that $\mathbf{dim} U = \mathbf{dim} E$. It follows that U decomposes into $U = U_P \oplus U_R$ where U_P is preprojective and U_R is regular. As $\partial_U = \partial_E = 0$, we have $U_P = 0$ and thus U_R is regular. Now $\mathbf{dim} U_R = \mathbf{dim} E$ but U_R is a regular submodule of M_F . By uniseriality of the regular components, U_R has to be indecomposable. As $\mathbf{dim} E$ is a real root, it follows that $E \simeq U_R \subset M_F$ and thus $F = E$. As the G_δ -orbit of M_E is not open, it follows that the value of $\chi(\mathrm{Gr}_{\mathbf{dim} E}(-))$ has to be zero on $U_{\delta, \mathbf{dim} E}$ and so $U_\delta \cap \mathcal{O}_{M_E} = \emptyset$ for any quasi-simple E in an exceptional tube. It follows that

$$U_\delta \cap \mathfrak{M}_\delta \subset \bigsqcup_{\lambda \in \mathbb{P}_0^1} \mathcal{O}_{M_\lambda}.$$

Thus, $X_\delta = X_{M_\lambda}$ for some $\lambda \in \mathbb{P}_0^1$ and lemma 3.14 implies that $X_\delta = X_{M_\lambda}$ for any $\lambda \in \mathbb{P}_0^1$.

Now if $n > 1$, the canonical decomposition of $n\delta$ is $\delta \oplus \cdots \oplus \delta$. Proposition 3.7 implies then that

$$X_{n\delta} = X_\delta^n = X_{M_\lambda}^n.$$

□

Remark 3.16. Note that it is not clear at this time that $X_{M_E} \neq X_{M_\lambda}$ if $\lambda \in \mathbb{P}_0^1$ and E is a quasi-simple in an exceptional tube. It will appear in theorem 3.25 that these values are in fact always different and that the difference can be completely described. Moreover, it will also turn out that if E and F are non-isomorphic quasi-simple modules taken in exceptional tubes, then $X_{M_E} \neq X_{M_F}$ if E and F do not belong both to tubes of rank 2.

3.2.3. Generic variables associated to positive real non-Schur roots. Now, it remains to compute $X_{\mathbf{d}}$ when \mathbf{d} is a real root which is not a Schur root. If \mathbf{d} is such a root, then according to Kac's theorem, there exists a unique indecomposable module M of dimension vector \mathbf{d} . If M is preprojective or preinjective, it is known that $\mathrm{End}_{kQ}(M) \simeq k$ and thus \mathbf{d} is a real Schur root. It follows that M has to be a regular module and $\delta \not\geq \mathbf{d}$ (see [CB92] for example). It is proved in [Kac82] that $\mathbf{d} = \delta \oplus \cdots \oplus \delta \oplus \mathbf{d}_0$ where \mathbf{d}_0 is the root of smallest height in $(\mathbf{d} + \mathbb{Z}\delta) \cap \Phi_{\geq 0}(Q)$. In particular, \mathbf{d}_0 is a real Schur root. The following proposition gives an explicit description of $X_{\mathbf{d}}$ in this case.

Proposition 3.17. *Let Q be an affine quiver, \mathbf{d} be a real root which is not a Schur root and write $\mathbf{d} = \delta^{\oplus n} \oplus \mathbf{d}_0$ its canonical decomposition. Then*

$$X_{\mathbf{d}} = X_{M_\lambda}^n X_{M_0}$$

for any $\lambda \in \mathbb{P}_0^1$ and M_0 being the unique indecomposable module of dimension vector \mathbf{d}_0 .

Proof. According to lemmas 3.12 and 3.15, $X_{\mathbf{d}_0} = X_{M_0}$ where M_0 is the unique indecomposable module in $\mathrm{rep}(Q, \mathbf{d}_0)$ and $X_{n\delta} = X_\delta^n = X_{M_\lambda}^n$ for any $\lambda \in \mathbb{P}_0^1$. Proposition 3.7 implies then that

$$X_{\mathbf{d}} = X_{n\delta} X_{\mathbf{d}_0} = X_{M_\lambda}^n X_{M_0}.$$

□

We can now give a complete description of the generic variables:

Proposition 3.18. *Let Q be an affine quiver and $\mathbf{d} \in \mathbb{Z}^{Q_0}$, denote by \mathcal{E} a set of representatives of isoclasses of regular modules M . Then*

$$\mathcal{B}'(Q) = \{\text{cluster monomials}\} \sqcup \left\{ X_{M_{\lambda}^{\oplus n} \oplus E} : n \geq 1, E \in \text{add } \mathcal{E} \text{ is rigid} \right\}$$

Proof. Fix $\mathbf{d} \in \mathbb{Z}^{Q_0}$, write $[\mathbf{d}]_+ = \delta^k \oplus \bigoplus_{i=1}^n e_i$ the canonical decomposition of $[\mathbf{d}]_+$. If $k \neq 0$, then as δ is sincere, we have $\mathbf{d} \in \mathbb{N}^{Q_0}$. According to proposition 3.5 and lemma 3.15, there is some M in $\mathfrak{M}_{\mathbf{d}} \cap U_{\mathbf{d}}$ such that

$$M = \bigoplus_{j=1}^k M_{\lambda_j} \oplus \bigoplus_{i=1}^n M_i$$

where the $\lambda_j \in \mathbb{P}_0^1$ are pairwise distinct and the M_i are indecomposable rigid modules such that $\text{End}_{kQ}(M_i) \simeq k$ and $\text{Ext}_{kQ}^1(M_i, M_l) = 0$ if $i \neq l$.

If one of the M_i is preprojective, then there is some vertex $v \in Q_0$ and some integer $s \geq 0$ such that $M_i \simeq \tau^{-s} P_v$ and then

$$\text{Ext}_{\mathcal{C}}^1(M_i, M_{\lambda_1}) = \text{Hom}_{\mathcal{C}}(P_v, M_{\lambda_1}) = \dim M_{\lambda_1}(v) = \delta_v = 1$$

which is a contradiction. Similarly, none of the M_i can be preinjective. It follows that each M_i is regular and thus the M_i are indecomposable regular modules in exceptional tubes such that $\dim M_i \leq \delta$. As $\text{Ext}_{kQ}^1(M_i, M_l) = 0$ for $i \neq l$, it follows that $\bigoplus_{i=1}^n M_i$ is regular rigid.

If $k = 0$, then $M \in \mathfrak{M}_{\mathbf{d}} \cap U_{\mathbf{d}}$ is rigid and thus $M \oplus \bigoplus_{d_i < 0} P_i[1]^{\oplus -d_i}$ is a rigid object. Thus $X_{\mathbf{d}}$ is a cluster monomial.

This proves the inclusion

$$\mathcal{B}'(Q) \subset \{\text{cluster monomials}\} \sqcup \left\{ X_{M_{\lambda}^{\oplus n} \oplus E} : n \geq 1, E \in \text{add } \mathcal{E} \text{ is rigid} \right\}.$$

We now prove the reverse inclusion. According to lemma 3.10 it suffices to prove that $\left\{ X_{M_{\lambda}^{\oplus n} \oplus E} : n \geq 1, E \in \text{add } \mathcal{E} \text{ is rigid} \right\} \subset \mathcal{B}'(Q)$. Assume that $E \in \text{add } \mathcal{E}$ is rigid. Then $X_{\dim E} = X_E$ by lemma 3.10. On the other hand, $X_{M_{\lambda}^{\oplus n}} = X_{n\delta}$ by lemma 3.15. Fix now $\lambda_1, \dots, \lambda_n$ pairwise distinct elements in \mathbb{P}_0^1 , as there are no extensions between the tubes we have

$$\text{Ext}_{\mathcal{C}_Q}^1(E, M_{\lambda_1} \oplus \dots \oplus M_{\lambda_n}) = 0$$

so $\text{Ext}_{\mathcal{C}_Q}^1(\dim E, n\delta)$ vanishes generally and by lemma 3.9, we have

$$X_{\dim E + n\delta} = X_{\dim E} X_{n\delta} = X_{E \oplus M_{\lambda}^{\oplus n}}$$

and this proves the proposition.

It remains to notice that the union is disjoint. Indeed, fix $X_{\mathbf{d}} = X_{M_{\lambda}^{\oplus n} \oplus E}$ with E rigid in $\text{add } \mathcal{E}$ and $n \geq 1$. Fix $\lambda_1, \dots, \lambda_n$ pairwise distinct elements in \mathbb{P}_0^1 , and decompose $E = E_1 \oplus \dots \oplus E_m$ into indecomposable summands. Then

$$M_{\lambda_1} \oplus \dots \oplus M_{\lambda_n} \oplus E_1 \oplus \dots \oplus E_m$$

is a direct sum of Schur representations such that $\text{Ext}_{kQ}^1(U, V) = 0$ for any two indecomposable direct summands U, V . It follows that

$$\mathbf{d} = \delta^{\oplus n} \oplus \bigoplus_{i=1}^n \dim E_i$$

is the canonical decomposition of \mathbf{d} . As $\dim \text{Ext}_{kQ}^1(M, M) \neq 0$ for any representation in $\text{rep}(Q, n\delta)$ for $n \geq 1$, there cannot be any rigid module in $\text{rep}(Q, \mathbf{d})$. The denominators theorem ensures then that there is no rigid object M in \mathcal{C}_Q such that $X_M = X_{\mathbf{d}}$. \square

3.3. Generic variables as generators in type \tilde{A} . Following the ideas of [CK08], in order to prove that $\mathcal{B}'(Q)$ is a \mathbb{Z} -basis in $\mathcal{A}(Q)$ when Q is affine of type \tilde{A} . We first prove that every X_M can be written as a \mathbb{Z} -linear combination of $X_{\mathbf{d}}$. We will do it in two steps: the first step will consist in a study of the values of X_M when M is indecomposable and the second step will be devoted to the case where M is decomposable.

Fix M an indecomposable object in \mathcal{C}_Q . If M is in the transjective component. Then M is rigid and thus lemma 3.10 implies that $X_M \in \mathcal{B}'(Q)$. Assume now that M is not in the transjective component. Then there are three possibilities:

- **dim** M is a real Schur root,
- **dim** M is an imaginary Schur root,
- **dim** M is a positive non-Schur root.

In the case **dim** M is a real Schur root, M is rigid and thus $X_M \in \mathcal{B}'(Q)$ by lemma 3.10. The other cases need a deeper study.

3.3.1. Modules from homogeneous tubes.

Lemma 3.19. *Let Q be a quiver of affine type, $\lambda \in \mathbb{P}_0^1$ and $n \geq 1$. Then*

$$X_{M_\lambda^{(n)}} \in \mathbb{Z}\mathcal{B}'(Q).$$

Proof. We first note that $X_{M_\lambda} = X_\delta \in \mathcal{B}'(Q)$. Now, if $n \geq 1$, it follows from corollary 3.13 that $X_{M_\lambda^{(n)}} = C_n(X_{M_\lambda})$. As C_n is a polynomial of degree n with integer coefficients, $X_{M_\lambda^{(n)}}$ is a \mathbb{Z} -linear combination of $X_{M_\lambda}^k$ for $k = 0, \dots, n$. According to lemma 3.15, $X_{M_\lambda}^n = X_{n\delta}$ for any $n \geq 1$ and then $X_{M_\lambda^{(n)}} \in \mathbb{Z}\mathcal{B}'(Q)$. \square

3.3.2. Modules from exceptional tubes. From now on, we focus on the case where Q is a quiver of affine type \tilde{A} which is not isomorphic to the Kronecker quiver. We fix a quasi-simple module E in an exceptional tube of rank $p > 1$. We denote by $M_E^{(n)}$ the unique indecomposable module of dimension vector $n\delta$ (or equivalently of quasi length np) and quasi-socle E . If M is indecomposable regular, we denote by $\text{q.rad}M$ its quasi-radical. We prove that for any $n \geq 1$, $X_{M_E^{(n)}} \in \mathbb{Z}\mathcal{B}'(Q)$. First, we prove the striking *difference property*

$$X_{M_E} = X_{M_\lambda} + X_{\text{q.rad}M_E/E}$$

for any $\lambda \in \mathbb{P}_0^1$. This will be one of the essential points of this article and require preliminary results.

In our study, grassmannians of submodules of the quasi-simple modules M_λ will be of great interest. The following lemma simplifies this study in the particular case of a quiver of type \tilde{A} .

Lemma 3.20. *Let Q be an acyclic quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector such that $d_i \leq 1$ for any $i \in Q_0$. Fix $M \in \text{rep}(Q, \mathbf{d})$ and $\mathbf{e} \leq \mathbf{d}$ another dimension vector. Then if $\text{Gr}_{\mathbf{e}}(M) \neq \emptyset$, it is a point. In particular, we have $\chi(\text{Gr}_{\mathbf{e}}(M)) = 1$.*

Proof. Assume that $\text{Gr}_{\mathbf{e}}(M) \neq \emptyset$ and fix $N \in \text{Gr}_{\mathbf{e}}(M)$. For every i such that $e_i \neq 0$, $N(i)$ is a non-zero subspace of the one-dimensional vector space $M(i)$ and then $N(i) = M(i)$. Thus, N is the representation given by

$$N(i) = \begin{cases} M(i) & \text{if } e_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $N(i \rightarrow j) = M(i \rightarrow j)|_{N(i)}$. Therefore, $\text{Gr}_{\mathbf{e}}(M) = \{N\}$ is a point. \square

The following theorem will be referred to as the *Schofield's theorem*. It is of a great use in the study of grassmannians of submodules.

Theorem 3.21 ([Sch92]). *Let Q be an acyclic quiver and $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0}$ such that $\text{Ext}_{kQ}^1(\mathbf{d}, \mathbf{e}) = 0$. Then any representation of dimension vector $\mathbf{d} + \mathbf{e}$ contains a sub-representation of dimension vector \mathbf{d} .*

Lemma 3.22. *Fix Q a quiver of affine type $\tilde{\mathbb{A}}$, E a quasi-simple module in an exceptional tube, $\lambda \in \mathbb{P}_0^1$ and \mathbf{v} a dimension vector. If $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$. Then $\text{Ext}_{kQ}^1(\mathbf{v}, \delta - \mathbf{v})$ vanishes generally and $\chi(\text{Gr}_{\mathbf{v}}(M_\lambda)) = \chi(\text{Gr}_{\mathbf{v}}(M_E))$.*

Proof. If $\mathbf{v} = \delta$, then $\text{Gr}_\delta(M_\lambda) = \{M_\lambda\}$ and $\text{Gr}_\delta(M_E) = \{M_E\}$ so

$$\chi(\text{Gr}_\delta(M_E)) = \text{Gr}_\delta(M_\lambda) = 1.$$

Assume now that $\mathbf{v} \neq \delta$ is such that $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$ and fix some proper submodule $V \subset M_\lambda$ of dimension vector \mathbf{v} . Then V is preprojective and M_λ/V is a direct sum of preinjective or regular modules. It follows that $\text{Ext}_{kQ}^1(V, M_\lambda/V) = 0$ and thus $\text{Ext}_{kQ}^1(\mathbf{v}, \delta - \mathbf{v}) = 0$. According to theorem 3.21 every representation of dimension δ contains a sub-representation of dimension \mathbf{v} . In particular, $\text{Gr}_{\mathbf{v}}(M_E) \neq \emptyset$. Now, lemma 3.20 implies that $\chi(\text{Gr}_\delta(M_E)) = \chi(\text{Gr}_\delta(M_\lambda)) = 1$ and the lemma is proved. \square

Proposition 3.23. *Fix Q a quiver of affine type $\tilde{\mathbb{A}}$, E a quasi-simple module in an exceptional tube, $\lambda \in \mathbb{P}_0^1$ and $V \subset M_E$ a proper submodule of dimension vector \mathbf{v} . The following holds:*

- (1) *If $E \not\subset V$, then $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$,*
- (2) *If $E \subset V \subset \text{q.rad} M_E$, then $\text{Gr}_{\mathbf{v}}(M_\lambda) = \emptyset$,*
- (3) *If $E \subset V \not\subset \text{q.rad} M_E$, then $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$.*

Proof. First we notice that if $V \subset M_E$, V does not have any preinjective summand. We thus write $V = V_P \oplus V_R$ with V_P preprojective and V_R regular. Note also that the uniseriality of the regular components force V_R to be indecomposable and to contain E if it is not zero.

If $E \not\subset V$, then $V_R = 0$ and $V = V_P$ is preprojective. M_E/V being a direct sum of regular and preinjective summands, it follows that $\text{Ext}_{kQ}^1(V, M_E/V) = 0$ and thus theorem 3.21 implies that $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$. This proves the first point.

If $E \subset V$, then V is not preprojective. Now M_E/V is a direct sum of regular and preinjective summands. It follows that

$$\text{Ext}_{kQ}^1(V, M_E/V) \simeq \text{Hom}_{kQ}(M_E/V, \tau V) = \text{Hom}_{kQ}(M_E/V, \tau V_R).$$

As $E \subset V \subset \text{q.rad} M_E$, we have

$$M_E/\text{q.rad} M_E \simeq (M_E/V)/(\text{q.rad} M_E/V)$$

and thus $\tau E \simeq M_E/\text{q.rad} M_E$ is a quotient of M_E/V . In particular the space $\text{Hom}_{kQ}(M_E/V, \tau E)$ is not reduced to zero. As $\tau E \subset \tau V_R$, we get

$$\text{Hom}_{kQ}(M_E/V, \tau V_R) \neq 0$$

and thus

$$\text{Ext}_{kQ}^1(V, M_E/V) \neq 0.$$

As $\delta_i = 1$ for every $i \in Q_0$, it follows that $\text{Hom}_{kQ}(V, M_E/V) = 0$ and thus

$$\langle \mathbf{v}, \delta - \mathbf{v} \rangle < 0.$$

In particular, for any two representations X and Y of respective dimensions \mathbf{v} and $\delta - \mathbf{v}$, we have $\text{Ext}_{kQ}^1(X, Y) \neq 0$ and thus $\text{Ext}_{kQ}^1(\mathbf{v}, \delta - \mathbf{v})$ does not vanish generally. It thus follows from lemma 3.22 that $\text{Gr}_{\mathbf{v}}(M_\lambda) = \emptyset$. This proves the second point.

Assume now that $E \subset V \not\subset \text{q.rad}M_E$. As $E \subset V$, $V_R \neq 0$. Now if $V_P = 0$, $V = V_R$ is a proper regular submodule of M_E , it is thus contained in its quasi-radical $\text{q.rad}M_E$, this is a contradiction and thus $V = V_P \oplus V_R$ is a non-trivial decomposition. We now prove that $\text{Ext}_{kQ}^1(V_R, V_P) \neq 0$. We denote by

$$\pi : M_E \longrightarrow M_E/\text{q.rad}M_E \simeq \tau E$$

the canonical projection. As $V \not\subset \text{q.rad}M_E$, $\pi(V) \neq 0$ and thus $\pi|_V : V \longrightarrow \tau E$ is a non-zero $V \longrightarrow \tau E$, thus

$$\text{Hom}_{kQ}(V, \tau E) \simeq \text{Hom}_{kQ}(V_P, \tau E) \oplus \text{Hom}_{kQ}(V_R, \tau E) \neq 0.$$

As $E \subset V_R \subset \text{q.rad}M_E$, the combinatorics of tubes proves that

$$\text{Hom}_{kQ}(V_R, \tau E) = 0$$

and thus $\text{Hom}_{kQ}(V_P, \tau E) \neq 0$. As $\tau E \subset \tau V_R$, it follows that $\text{Hom}_{kQ}(V_P, \tau V_R) \neq 0$ and thus $\text{Ext}_{kQ}^1(V_R, V_P) \neq 0$.

Fix now a non-split short exact sequence

$$(3) \quad 0 \longrightarrow V_P \xrightarrow{\iota} X \xrightarrow{p} V_R \longrightarrow 0,$$

we prove that X is preprojective. Applying the functor $\text{Hom}_{kQ}(-, M_\lambda)$ to the sequence, we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{kQ}(V_R, M_\lambda) &\longrightarrow \text{Hom}_{kQ}(X, M_\lambda) \longrightarrow \text{Hom}_{kQ}(V_P, M_\lambda) \\ &\longrightarrow \text{Ext}_{kQ}^1(V_R, M_\lambda) \longrightarrow \text{Ext}_{kQ}^1(X, M_\lambda) \longrightarrow \text{Ext}_{kQ}^1(V_P, M_\lambda) \longrightarrow 0 \end{aligned}$$

As there are neither morphisms nor extensions between the tubes, we get an isomorphism

$$\text{Hom}_{kQ}(X, M_\lambda) \simeq \text{Hom}_{kQ}(V_P, M_\lambda).$$

More precisely, every morphism $V_P \longrightarrow M_\lambda$ factorises in $V_P \xrightarrow{\iota} X \longrightarrow M_\lambda$. Decompose $X = X_P \oplus X_R \oplus X_I$ into preprojective, regular and preinjective direct summands. As \mathbb{P}_0^1 is infinite, we can assume that λ is chosen such that $V_R \notin \mathcal{T}_\lambda$ and thus $\text{Hom}_{kQ}(X_R, M_\lambda) = 0$. Consider the inclusion $f : V_P \longrightarrow M_\lambda$. Then there is a factorization

$$V_P \xrightarrow{[i_1, i_2, i_3]^t} X_P \oplus X_R \oplus X_I \xrightarrow{[g_1, 0, 0]} M_\lambda$$

and thus $f = g_1 i_1$ factorizes through i_1 . In particular, i_1 is injective and thus $V_P \subset X_P$. Also, we have $\dim X = \dim V \leq \delta$ so $\dim X(i) \leq 1$ for all i . It follows that $\text{Hom}_{kQ}(V_P, X_R) = \text{Hom}_{kQ}(V_P, X_I) = 0$. Thus $i_2 = i_3 = 0$ and then the sequence (3) gives

$$V_R \simeq X/V_P \simeq X_P/V_P \oplus X_R \oplus X_I.$$

Now V_R is regular so $X_I = 0$. It follows that $\partial_{X_P} = \partial_X = \partial_V = \partial_{V_P}$ and thus X_P/V_P is a regular module. By uniseriality, V_R is indecomposable and thus either X_P/V_P is zero, or X_R is zero. If X_P/V_P is zero, the sequence (3) splits, which is a contradiction. It follows that $X_R = 0$ and thus that $X = X_P$ is preprojective.

As M_E/V is a direct sum of regular and preinjective summands, we thus have

$$\text{Ext}_{kQ}^1(X, M_E/V) = 0$$

and thus $\text{Ext}_{kQ}^1(\mathbf{v}, \delta - \mathbf{v})$ vanishes generally. By theorem 3.21, we thus have $\text{Gr}_{\mathbf{v}}(M_\lambda) \neq \emptyset$ and this proves the last point. \square

Definition 3.24. We say that a quiver Q of affine type satisfies the *difference property* if for every quasi-simple module E in an exceptional tube, the following identity holds

$$(4) \quad X_{M_E} = X_{M_\lambda} + X_{\text{q.rad}M_E/E}$$

for any $\lambda \in \mathbb{P}_0^1(Q)$.

Theorem 3.25. *A quiver Q of affine type $\tilde{\mathbb{A}}$ satisfies the difference property.*

Proof. For any dimension vector \mathbf{v} , proposition 3.23 and lemma 3.20 imply that

$$\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) \neq \emptyset \Leftrightarrow \mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E) = \emptyset$$

where $\mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E)$ denotes the modules $M \in \mathrm{Gr}_{\mathbf{v}}(\mathrm{q.rad} M_E)$ such that $E \subset M$. Moreover, if $\mathrm{Gr}_{\mathbf{v}}(M_E) \neq \emptyset$ and $\mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E) = \emptyset$, we have $\chi(\mathrm{Gr}_{\mathbf{v}}(M_{\lambda})) = \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) = 1$ and if $\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) \neq \emptyset$, we have $\chi(\mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E)) = \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) = 1$.

We write $\mathbf{e} = \dim E$, we have

$$\begin{aligned} X_{M_E} &= \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &= \sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) \neq \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &\quad + \sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &= \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}}(M_{\lambda})) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &\quad + \sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &= X_{M_{\lambda}} + \sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \end{aligned}$$

Now, we have

$$\sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} = \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle}$$

As

$$\begin{cases} \mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E) & \longrightarrow & \mathrm{Gr}_{\mathbf{v}-\mathbf{e}}(\mathrm{q.rad} M_E/E) \\ U & \mapsto & U/E \end{cases}$$

is an isomorphism, it follows that

$$\sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} = \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}-\mathbf{e}}(\mathrm{q.rad} M_E/E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle}$$

But if we write $\mathbf{n} = \dim \mathrm{q.rad}(M_E/E) = \delta - c(\mathbf{e}) - \mathbf{e}$ where c is the Coxeter transformation, we compute

$$\begin{aligned} -\langle (\mathbf{v} - \mathbf{e}), \alpha_i \rangle - \langle \alpha_i, \mathbf{n} - (\mathbf{v} - \mathbf{e}) \rangle &= -\langle \mathbf{v} - \mathbf{e}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{e} - c(\mathbf{e}) - \mathbf{v} + \mathbf{e} \rangle \\ &= -\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle + \langle \mathbf{e}, \alpha_i \rangle + \langle \alpha_i, c(\mathbf{e}) \rangle \\ &= -\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle \end{aligned}$$

and thus

$$\begin{aligned} X_{\mathrm{q.rad} M_E/E} &= \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}-\mathbf{e}}(\mathrm{q.rad} M_E/E)) \prod_i u_i^{-\langle (\mathbf{v}-\mathbf{e}), \alpha_i \rangle - \langle \alpha_i, \mathbf{n} - (\mathbf{v}-\mathbf{e}) \rangle} \\ &= \sum_{\mathbf{v}} \chi(\mathrm{Gr}_{\mathbf{v}-\mathbf{e}}(\mathrm{q.rad} M_E/E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \\ &= \sum_{\mathrm{Gr}_{\mathbf{v}}(M_{\lambda}) = \emptyset} \chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) \prod_i u_i^{-\langle \mathbf{v}, \alpha_i \rangle - \langle \alpha_i, \delta - \mathbf{v} \rangle} \end{aligned}$$

This implies that

$$X_{M_E} = X_{M_\lambda} + X_{\text{q.rad}M_E/E}$$

□

Corollary 3.26. *Let Q be a quiver of affine type satisfying the difference property. If E is a quasi-simple module contained in a tube of rank $p > 2$. Then $X_{M_E} \neq X_{M_F}$ for any quasi-simple module $F \neq E$ in an exceptional tube.*

Proof. As E is contained in an exceptional tube of rank $p > 2$, it follows that $\text{q.rad}M_E$ is of quasi-length $l > 1$ and thus $\text{q.rad}M_E/E$ is a non-zero indecomposable rigid module. Fix F a quasi-simple not isomorphic to E . Then $\text{q.rad}M_F/F$ is an indecomposable rigid module which is not isomorphic to $\text{q.rad}M_E/E$. It follows that $\dim \text{q.rad}M_F/F \neq \dim \text{q.rad}M_E/E$ and thus the denominators theorem implies that $X_{\text{q.rad}M_F/F} \neq X_{\text{q.rad}M_E/E}$. Thus,

$$X_{M_E} = X_{M_\lambda} + X_{\text{q.rad}M_E/E} \neq X_{M_\lambda} + X_{\text{q.rad}M_F/F} = X_{M_F}.$$

□

Remark 3.27. It is conjectured that every quiver of affine type satisfies the difference property.

Example 3.28. The most simple case of affine quivers after the Kronecker quiver are the quivers of affine type $\hat{\mathbb{A}}_{2,1}$. They are all isomorphic to

$$Q : \begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \xrightarrow{\quad} & 3 \end{array}$$

The Auslander-Reiten quiver of $kQ\text{-mod}$ contains only one exceptional tube of rank 2. The quasi-simple modules of this tube are given by

$$E_0 : \begin{array}{ccc} & 0 & \\ \nearrow 0 & & \searrow \\ k & \xrightarrow{1} & k \end{array}$$

and

$$E_1 \simeq S_2 : \begin{array}{ccc} & k & \\ \nearrow & & \searrow 0 \\ 0 & \xrightarrow{\quad} & 0 \end{array}$$

satisfying $\tau E_1 \simeq E_0$ and $\tau E_0 \simeq E_1$. The corresponding indecomposable modules of dimension δ are:

$$M_{E_0} : \begin{array}{ccc} & k & \\ \nearrow 0 & & \searrow 1 \\ k & \xrightarrow{1} & k \end{array}$$

and

$$M_{E_1} : \begin{array}{ccc} & k & \\ \nearrow 1 & & \searrow 0 \\ k & \xrightarrow{1} & k. \end{array}$$

For any $0 \neq \lambda \in k$, we set

$$M_\lambda : \begin{array}{ccc} & k & \\ 1 \nearrow & & \searrow \lambda \\ k & \xrightarrow{1} & k \end{array}$$

and we set

$$M_\infty : \begin{array}{ccc} & k & \\ 1 \nearrow & & \searrow 1 \\ k & \xrightarrow{0} & k \end{array}$$

We identify \mathbb{P}^1 and $k \sqcup \{\infty\}$. Then the M_λ for $\lambda \in \mathbb{P}^1 \setminus \{0\}$ are quasi-simple representations in homogeneous tubes and $\mathbb{P}_0^1(Q) = \mathbb{P}^1 \setminus \{0\}$.

The following table sums up the situation where $\lambda \in \mathbb{P}_0^1(Q)$:

e	[000]	[001]	[010]	[100]	[011]	[101]	[110]	[111]
$\text{Gr}_e(M_\lambda)$	0	S_3	\emptyset	\emptyset	P_2	\emptyset	\emptyset	M_λ
$\chi(\text{Gr}_e(M_\lambda))$	1	1	0	0	1	0	0	1
$\text{Gr}_e(M_{E_0})$	0	S_3	\emptyset	\emptyset	P_2	E_0	\emptyset	M_{E_0}
$\chi(\text{Gr}_e(M_{E_0}))$	1	1	0	0	1	1	0	1
$\text{Gr}_e(M_{E_1})$	0	S_3	E_1	\emptyset	$S_2 \oplus S_3$	\emptyset	E_1	M_{E_1}
$\chi(\text{Gr}_e(M_{E_1}))$	1	1	1	0	1	0	0	1

It is important to note that even if the grassmannians of dimension [011] have the same Euler characteristic, they are not equal. More precisely,

$$\text{Gr}_{[011]}(M_{E_1}) = \{S_2 \oplus S_3\}$$

whereas

$$\text{Gr}_{[011]}(M_\lambda) = \text{Gr}_{[011]}(M_{E_0}) = \{P_2\}.$$

With the notations of proposition 3.23, if $V = S_2 \oplus S_3$, we have $V_P = S_3$, $V_R = S_2$ and there is a non-trivial extension

$$0 \longrightarrow S_3 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0,$$

illustrating the second point of proposition 3.23.

Now, if we compute the generalized variables corresponding to these modules, we find:

$$X_{M_{E_0}} = X_{M_{E_1}} = \frac{u_2 u_1^2 + u_3 u_1 u_2 + u_1 + u_3 + u_2 u_3^2}{u_1 u_2 u_3}$$

and

$$X_{M_\lambda} = \frac{u_2 u_1^2 + u_1 + u_3 + u_2 u_3^2}{u_1 u_2 u_3} = X_{M_{E_1}} - 1.$$

As $\text{q.rad} M_{E_i} = E_i$ for $i = 1, 2$, it follows that $\text{q.rad} M_{E_i}/E_i = 0$, this illustrates theorem 3.25.

Remark 3.29. Note that the tubes of rank 2 give nice examples of modules giving the same generalized variables but having different characteristics of grassmannians.

Example 3.30. Consider the quiver of affine type $\tilde{A}_{3,1}$:

$$Q : \begin{array}{ccc} & 2 & \longrightarrow 3 \\ & \nearrow & \searrow \\ 1 & \xrightarrow{\quad} & 4 \end{array}$$

For any $\lambda \in k$, we set

$$M_\lambda : \begin{array}{ccccc} & & k & \xrightarrow{1} & k \\ & \nearrow 1 & & & \searrow \lambda \\ k & & & \xrightarrow{1} & k \end{array}$$

and

$$M_\infty : \begin{array}{ccccc} & & k & \xrightarrow{1} & k \\ & \nearrow 1 & & & \searrow 1 \\ k & & & \xrightarrow{0} & k \end{array}$$

We identify \mathbb{P}^1 and $k \sqcup \{\infty\}$. Then, for any $\lambda \in \mathbb{P}^1(k)$, M_λ is an indecomposable representation of dimension δ and moreover, if $\lambda \neq 0$, then M_λ belongs to an homogeneous tube and $\mathbb{P}_0^1(Q) = \mathbb{P}^1 \setminus \{0\}$.

The representation

$$E_0 : \begin{array}{ccccc} & & 0 & \longrightarrow & k \\ & \nearrow & & & \searrow 0 \\ 0 & & & \longrightarrow & 0 \end{array}$$

is quasi-simple and, is given by

$$E_1 = \tau^{-1}E_0 : \begin{array}{ccccc} & & k & \xrightarrow{0} & 0 \\ & \nearrow & & & \searrow \\ 0 & & & \longrightarrow & 0 \end{array}$$

$$E_2 = \tau^{-2}E_0 : \begin{array}{ccccc} & & 0 & \longrightarrow & 0 \\ & \nearrow 0 & & & \searrow \\ k & & & \xrightarrow{1} & k \end{array}$$

and $\tau^{-3}E_0 = E_0$. Thus E_0 is contained in an exceptional tube of rank 3 and moreover $M_0 = M_{E_0}$.

The tube is thus

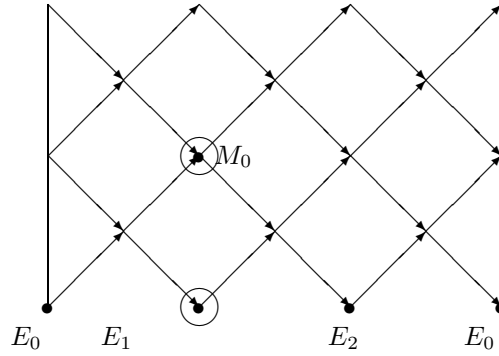


FIGURE 2. The exceptional tube of $\tilde{A}_{3,1}$

Note that $\text{q.rad} M_0 = E_0^{(2)}$ and $\text{q.rad} M_0/E_0 = E_1$. We compute the corresponding generalized variable and we get

$$X_{E_1} = \frac{u_1 + u_3}{u_2}$$

We now compute the grassmannians for some $\lambda \in \mathbb{P}_0^1$. We only write dimension vectors for which one of the two considered grassmannians is non-empty:

e	[0000]	[0001]	[0010]	[0110]	[0011]	[0111]	[1111]
$\text{Gr}_{\mathbf{e}}(M_\lambda)$	0	S_4	\emptyset	\emptyset	P_3	P_2	M_λ
$\chi(\text{Gr}_{\mathbf{e}}(M_\lambda))$	1	1	0	0	1	1	1
$\text{Gr}_{\mathbf{e}}(M_0)$	0	S_4	E_0	$E_0^{(2)}$	$E_0 \oplus S_4$	$E_0^{(2)} \oplus S_4$	M_0
$\chi(\text{Gr}_{\mathbf{e}}(M_{E_0}))$	1	1	1	1	1	1	1

A direct computation gives:

$$X_{M_0} = \frac{u_1 u_4 u_3^2 + u_3 u_1^2 u_4 + u_3 u_2 u_4^2 + u_4 u_3 + u_3 u_2 u_1^2 + u_1 u_4 + u_2 u_1}{u_1 u_2 u_3 u_4}$$

and

$$X_{M_\lambda} = \frac{u_3 u_2 u_1^2 + u_2 u_1 + u_1 u_4 + u_4 u_3 + u_3 u_2 u_4^2}{u_1 u_2 u_3 u_4}.$$

Computing the difference, we get

$$X_{M_0} - X_{M_\lambda} = \frac{u_1 + u_3}{u_2} = X_{E_1},$$

this illustrates theorem 3.25.

It is interesting to note that for $\mathbf{e} = [0011]$, $V \in \text{Gr}_{\mathbf{e}}(M_0)$ is the representation $E_0 \oplus S_4$. Using the notations of proposition 3.23, $V_R = E_0$, $V_P = S_4$ and $P_3 \in \text{Gr}_{\mathbf{e}}(M_\lambda)$ is indeed the central term of an extension

$$0 \longrightarrow S_4 \longrightarrow P_3 \longrightarrow E_0 \longrightarrow 0,$$

illustrating proposition 3.23.

Corollary 3.31. *Fix Q a quiver of affine type satisfying the difference property, E a quasi-simple module in an exceptional tube. Then $X_{M_E} \in \mathbb{Z}\mathcal{B}'(Q)$.*

Proof. If we denote by p the rank of the exceptional tube containing E , $\text{q.rad} M_E/E$ is an indecomposable regular module of quasi-length $p-2$. If $p = 2$, then $\text{q.rad} M_E/E$ is zero and thus equality (4) implies that

$$X_{M_E} = X_{M_\lambda} + 1 \in \mathbb{Z}\mathcal{B}'(Q).$$

If $p > 2$, it follows that $\mathbf{dim} E$ is a Schur root and that $X_{\text{q.rad} M_E/E} = X_{\mathbf{dim} \text{q.rad} M_E/E} \in \mathcal{B}'(Q)$. Now, we know that $X_{M_\lambda} = X_\delta \in \mathcal{B}'(Q)$ so equality (4) implies that $X_{M_E} \in \mathbb{Z}\mathcal{B}'(Q)$. \square

We now want to prove that $X_{M_E^{(n)}}$ is generated by $\mathcal{B}'(Q)$ over \mathbb{Z} for any $n \geq 2$ and any quasi-simple E in an exceptional tube.

Proposition 3.32. *Let Q be a quiver of affine type satisfying the difference property and \mathcal{T} be an exceptional tube in the AR quiver of Q . Then $X_M \in \mathbb{Z}\mathcal{B}'(Q)$ for any module M in $\text{add } \mathcal{T}$.*

Proof. We prove it by induction on the dimension vector of M . If M is quasi-simple, then $\mathbf{dim} M$ is a real Schur root and $X_M \in \mathcal{B}'(Q)$. Assume now that M is not quasi-simple and that for any N in \mathcal{T} such that $\mathbf{dim} N \leq \mathbf{dim} M$, we have $X_N \in \mathbb{Z}\mathcal{B}'(Q)$. We denote by p the rank of \mathcal{T} and E_0, \dots, E_{p-1} the quasi-simple modules of \mathcal{T} ordered such that $\tau E_i = E_{i-1}$ for every $i \in \mathbb{Z}/p\mathbb{Z}$.

Assume first that $M = M_E^{(n)}$ for some quasi-simple module E in \mathcal{T} . From equality (4), we have

$$X_{M_E} = X_{M_\lambda} + X_{\mathbf{q}, \text{rad} M_E / E}$$

Thus, by induction,

$$X_{M_E}^n - X_{M_\lambda}^n \in \mathbb{Z}\mathcal{B}'(Q)$$

but $X_{M_\lambda}^n = X_{n\delta} \in \mathcal{B}'(Q)$ so $X_{M_E}^n \in \mathbb{Z}\mathcal{B}'(Q)$.

Now, we know from [Dup09] that

$$X_{M_E^{(n)}} = P_{np,p}(X_{E_0}, \dots, X_{E_{p-1}})$$

where $P_{np,p}$ denotes the np -th generalized Chebyshev polynomial of rank p . In particular

$$X_{M_E^{(n)}} = X_{E_0 \oplus \dots \oplus E_{p-1}}^n + \sum_Y s_{Y,n} X_Y$$

where $s_{Y,n}$ are integers and Y runs over modules of the form $\bigoplus_{i=0}^{p-1} E_i^{\oplus n_i}$ with $\sum n_i \dim E_i \leq n\delta$. In particular,

$$X_{M_E} \in X_{E_0 \oplus \dots \oplus E_{p-1}} + \sum_Y s_{Y,1} X_Y$$

and thus

$$X_{M_E}^n \in X_{E_0 \oplus \dots \oplus E_{p-1}}^n + \sum_Y t_Y X_Y$$

where the t_Y are integers and Y run over modules of the form $\bigoplus_{i=0}^{p-1} E_i^{\oplus n_i}$ with $\sum n_i \dim E_i \leq n\delta$. It follows that

$$X_{M_E^{(n)}} = X_{M_E}^n + \sum_Y k_Y X_Y$$

where the k_Y are integers and Y run over modules of the form $\bigoplus_{i=0}^{p-1} E_i^{\oplus n_i}$ with $\sum n_i \dim E_i \leq n\delta$. As $X_{M_E}^n \in \mathbb{Z}\mathcal{B}'(Q)$ by the above discussion and all the X_Y are also in $\mathbb{Z}\mathcal{B}'(Q)$ by induction, we get $X_{M_E^{(n)}} \in \mathbb{Z}\mathcal{B}'(Q)$ for every $n \geq 1$.

Now assume that M is indecomposable but not of the form $M_E^{(n)}$. Then $\dim M$ is a real root. If $\dim M \leq \delta$, then $\dim M$ is a real Schur root and $X_M \in \mathcal{B}'(Q)$. We can thus assume that $\delta \leq \dim M$. More precisely, $\dim M = \delta^{\oplus n} \oplus \beta$ where β is a real Schur root. We write N the unique indecomposable of dimension vector β , it belongs to \mathcal{T} and we have an extension

$$0 \longrightarrow N \longrightarrow M \longrightarrow M_E^{(n)} \longrightarrow 0$$

for some quasi-simple module E in \mathcal{T} . Note that

$$\dim \text{Ext}_{kQ}^1(N, M_E^{(n)}) = \dim \text{Hom}_{kQ}(M_E^{(n)}, \tau N) = 1,$$

so it follows from the multiplication formula (see also [Dup09]), that

$$X_{M_E^{(n)}} X_N = X_M + X_B$$

where $B \simeq \ker f \oplus \text{coker } f[-1]$ for some morphism $f : M_E^{(n)} \rightarrow \tau N$. In particular $\dim B \leq \dim N + n\delta = \dim M$ and then, by induction $X_B \in \mathbb{Z}\mathcal{B}'(Q)$. Now, the above discussion proves that

$$X_{M_E^{(n)}} = X_{M_\lambda}^n + \sum_Y r_Y X_Y$$

where the r_Y are integers and Y runs over modules in $\text{add } \mathcal{T}$ such that $\dim Y \leq n\delta$. In particular,

$$\begin{aligned} X_{M_E^{(n)}} X_N &= X_{M_\lambda}^n X_N + \sum_Y r_Y X_{Y \oplus N} \\ &\in X_{n\delta} X_\beta + \mathbb{Z}\mathcal{B}'(Q) \\ &\in X_{n\delta+\beta} + \mathbb{Z}\mathcal{B}'(Q) \\ &\in \mathbb{Z}\mathcal{B}'(Q) \end{aligned}$$

and thus $X_M = X_{M_E^{(n)}} X_N - X_B \in \mathbb{Z}\mathcal{B}'(Q)$.

Now assume that M is decomposable. Then $M = M_1 \oplus M_2$ for some regular objects M_1, M_2 . But according to proposition 2.14, we can write X_M as a linear combination of X_Y where $Y = \bigoplus_{i=1}^n Y_i$ (indecomposable decomposition) is in $\text{add } \mathcal{T}$ such that $\text{Ext}_{\mathcal{C}_Q}^1(Y_i, Y_j) = 0$ for $i \neq j$. We prove by induction on the dimension vector that X_M can be written as a \mathbb{Z} -linear combination of X_V where V is of the form $M_\lambda^k X_R$ where $R \in \text{add } \mathcal{T}$ is rigid, $\lambda \in \mathbb{P}_0^1$ and $k \geq 0$. In particular, each X_V will belong to $\mathcal{B}'(Q)$ and thus $X_M \in \mathbb{Z}\mathcal{B}'(Q)$.

Fix some integer $1 \leq i \leq n$. The indecomposable summand Y_i is of the form $E_i^{(n_i)}$ for some integer $n_i \geq 0$ and some quasi-simple module E_i . Let denote by p the rank of the tube \mathcal{T} and write $n_i = k_i p + r_i$ with $0 \leq r_i < p$ the euclidean division of n_i by p . By theorem 2.8, we can write

$$X_{Y_i} = X_{E_i^{(r_i)}} X_{M_{F_i}^{(k_i)}} + X_B$$

with $\dim B \leq \dim Y_i$ and F_i a quasi-simple in \mathcal{T} . Now, thanks to the difference property $X_{M_{F_i}^{(k_i)}}$ can be written

$$X_{M_{F_i}^{(k_i)}} = X_{M_\lambda}^{k_i} + \sum_{\dim Z_i \leq k_i \delta} r_{Z_i} X_{Z_i}$$

where $Z_i \in \text{add } (\mathcal{T})$.

Then

$$X_Y = \prod_{i=1}^n X_{E_i}^{(r_i)} X_{M_\lambda}^{k_i} + \sum_{\dim U \leq \dim Y} s_U X_U$$

with $U \in \text{add } (\mathcal{T})$. By induction on the dimension vector, each X_U can be written as the expected \mathbb{Z} -linear combination.

Now

$$\prod_{i=1}^n X_{E_i}^{(r_i)} X_{M_\lambda}^{k_i} = X_{M_\lambda}^{\sum_{i=1}^n k_i} X_{\bigoplus_{j=1}^n E_j^{(r_j)}}.$$

If $k_j = 0$ for every $j = 1, \dots, n$, then each Y_j is rigid and thus Y is rigid and the result holds. Otherwise, $\dim \bigoplus_{j=1}^n E_j^{(r_j)} \leq \dim Y$ and thus, by induction, $X_{\bigoplus_{j=1}^n E_j^{(r_j)}}$ can be written as a \mathbb{Z} -linear combination of $X_{M_\lambda}^k X_R$ with R rigid in $\text{add } \mathcal{T}$ and thus $\prod_{i=1}^n X_{E_i}^{(r_i)} X_{M_\lambda}^{k_i}$ can be written as a \mathbb{Z} -linear combination of $X_{M_\lambda}^{k+\sum_{j=1}^n k_j} X_R$, which proves the assertion. \square

Corollary 3.33. *Let Q be a quiver of affine type satisfying the difference property. Then for any indecomposable object M in \mathcal{C}_Q , we have $X_M \in \mathbb{Q}\mathcal{B}'(Q)$.*

Lemma 3.34. *Let Q be an affine quiver satisfying the difference property and $Y = \bigoplus_i Y_i$ be an object in \mathcal{C}_Q such that $\text{Ext}_{\mathcal{C}}^1(Y_i, Y_j) = 0$ for any $i \neq j$. Then $X_Y \in \mathbb{Q}\mathcal{B}'(Q)$.*

Proof. If all the indecomposable direct summands of Y are rigid, then Y is also rigid and thus $X_Y \in \mathcal{B}'(Q)$. We thus assume that Y is not rigid.

Write

$$Y = Y_P \oplus \bigoplus_{\lambda \in \mathbb{P}^1} Y_\lambda$$

where Y_P has all its direct summands in the transjective component and for any λ in \mathbb{P}^1 , Y_λ has its direct summands in the tube \mathcal{T}_λ .

If $Y_\lambda = 0$ for every $\lambda \in \mathbb{P}^1$, then $Y = Y_P$ is rigid, thus we can assume that $Y_\lambda \neq 0$ for some $\lambda \in \mathbb{P}^1$ and that Y_λ has an indecomposable direct summand M such that $\text{Ext}_{kQ}^1(M, M) \neq 0$.

Assume that $Y_P \neq 0$ and fix an indecomposable direct summand N of Y_P . Then $N = P_i[s]$ for some $i \in Q_0$ and $s \in \mathbb{Z}$. As M is an indecomposable regular non-rigid object, it follows that $\mathbf{dim} M$ is sincere and so is $\mathbf{dim} \tau^{1-s} M \simeq M[1-s]$.

$$\begin{aligned} 0 &\neq \dim \tau^{1-s} M(i) \\ &= \dim \text{Hom}_{kQ}(P_i, \tau^{1-s} M) \\ &\leq \dim \text{Hom}_C(P_i, M[1-s]) \\ &= \dim \text{Hom}_C(P_i[s], M[1]) \\ &= \dim \text{Ext}_C^1(M, P_i[s]). \end{aligned}$$

so that $\text{Ext}_C^1(M, Y_P) \neq 0$ which is a contradiction. Thus, $Y_P = 0$ and

$$Y = \bigoplus_{\lambda \in \mathbb{P}^1} Y_\lambda$$

We now prove by induction on the dimension vector of Y that $X_Y \in \mathbb{Q}\mathcal{B}'(Q)$. If Y is a quasi-simple module, then either $Y = M_\mu$ for some $\mu \in \mathbb{P}_0^1$ or Y is quasi-simple in an exceptional tube and is thus rigid. In both cases $X_Y \in \mathcal{B}'(Q)$.

We now return to the general case $Y = \bigoplus_{\lambda \in \mathbb{P}^1} Y_\lambda$. If $Y_\lambda = 0$ for all $\lambda \in \mathbb{P}^1 \setminus \mathbb{P}_0^1$, then it follows from corollary 3.13 that X_{Y_λ} is a polynomial in X_δ . Thus, X_Y is also a polynomial in X_δ and thus as $X_{n\delta} = X_\delta^n$ for every $n \geq 1$, X_Y is in $\mathbb{Z}\mathcal{B}'(Q)$.

From now on, we assume that $Y_\lambda \neq 0$ for some $\lambda \in \mathbb{P}^1 \setminus \mathbb{P}_0^1$.

We claim that

$$X_{Y_\lambda} = X_\delta^{n_\lambda} X_{R_\lambda} + \sum_{\mathbf{dim} Z \leq \mathbf{dim} Y_\lambda} r_Z X_Z$$

for some integer $n_\lambda \geq 0$, some rigid module $R_\lambda \in \text{add } T_\lambda$ and some rational numbers r_Z indexed by modules $Z \in \text{add } T_\lambda$. We prove it by induction on the dimension vector of Y_λ .

If Y_λ is quasi-simple, then $R_\lambda = Y_\lambda$ gives the result. Assume now that Y_λ is any indecomposable module in \mathcal{T}_λ . Then it follows from the proof of 3.32 that

$$X_Y = X_{M_E^{(n)}} X_N - X_B$$

for some quasi-simple E in \mathcal{T}_λ , some integer $n \geq 1$ and some regular module B such that $\mathbf{dim} B \leq \mathbf{dim} Y$. In particular

$$X_Y = X_{M_\lambda}^n X_N + \sum r_Z X_Z$$

where Z runs over objects in \mathcal{T}_λ such that $\mathbf{dim} Z \leq \mathbf{dim} Y$. This proves the claim for Y_λ indecomposable.

Assume now that Y_λ is decomposable. Then each indecomposable direct summand of Y_λ satisfies the claim. Using inductively proposition 2.14, we prove the claim for any $Y_\lambda \in \mathcal{T}_\lambda$.

As for any $\lambda \in \mathbb{P}_0^1$, X_{Y_λ} is a polynomial in X_δ , it suffices to prove that

$$X_\delta^n \prod_{\lambda \in \mathbb{P}^1 \setminus \mathbb{P}_0^1} X_{Y_\lambda} \in \mathbb{Z}\mathcal{B}'(Q)$$

for any $\lambda \in \mathbb{P}_0^1$ and for $n \leq n_0$ where $n_0\delta = \mathbf{dim} \bigoplus_{\mu \in \mathbb{P}_0^1} Y_\mu$. We have

$$\begin{aligned} X_\delta^n \prod_{\lambda \in \mathbb{P}^1 \setminus \mathbb{P}_0^1} X_{Y_\lambda} &= X_\delta^n \prod_{\lambda \in \mathbb{P}^1 \setminus \mathbb{P}_0^1} \left(X_\delta^{n_\lambda} X_{R_\lambda} + \sum_{\mathbf{dim} Z \leq \mathbf{dim} Y_\lambda} r_Z X_Z \right) \\ &= X_\delta^{n + \sum_\lambda n_\lambda} X_{\bigoplus_\lambda R_\lambda} + \sum_{\mathbf{dim} Z \leq n\delta + \sum_\lambda Y_\lambda} r_Z X_Z \end{aligned}$$

where the λ run over $\mathbb{P}^1 \setminus \mathbb{P}_0^1$. Note that for any $n \leq n_0$, we have

$$\mathbf{dim} Z \leq n\delta + \sum_\lambda Y_\lambda \leq \mathbf{dim} Z \leq n_0\delta + \sum_\lambda Y_\lambda = \mathbf{dim} Y.$$

By induction on proposition 2.14, we can moreover assume that all the X_Z satisfy the hypothesis of the lemma. Thus, by induction, it follows that all the X_Z occurring in the last sum are in $\mathbb{Q}\mathcal{B}'(Q)$.

As there are no extensions between the tubes, $\bigoplus_\lambda R_\lambda$ is rigid and thus it follows from proposition 3.18 (and also from lemma 3.9) that

$$X_\delta^{n + \sum_\lambda n_\lambda} X_{\bigoplus_\lambda R_\lambda} \in \mathbb{Q}\mathcal{B}'(Q).$$

□

We are now able to prove that generic variables generate the cluster algebra in affine type \tilde{A} .

Proposition 3.35. *Let Q be a quiver of affine type satisfying the difference property. Then every element in $\mathcal{A}(Q)$ is in the \mathbb{Q} -vector space generated by $\mathcal{B}'(Q)$. Moreover, every element of $\mathcal{B}'(Q)$ belongs to $\mathcal{A}(Q)$.*

Proof. We first prove that $\mathcal{B}'(Q) \subset \mathcal{A}(Q)$. If Q is the Kronecker quiver, this is proved in [CZ06]. Assume now that Q is not the Kronecker quiver, it thus contains an exceptional tube.

Fix \mathbf{d} a dimension vector. If $\mathbf{d} \in \mathbb{Z}_{\leq 0}^{Q_0}$, $X_{\mathbf{d}}$ is a product of u_i and is thus in $\mathcal{A}(Q)$. As $X_{\mathbf{d}} = X_{[\mathbf{d}]_+} X_{[\mathbf{d}]_-}$, it suffices to prove the result for $\mathbf{d} \in \mathbb{N}^{Q_0}$.

If \mathbf{d} is a real Schur root, then there exists an indecomposable rigid module $M \in \text{rep}(Q, \mathbf{d})$ and thus theorem 1.3 implies X_M is a cluster variable.

If $\mathbf{d} = \delta$, then $X_{\mathbf{d}} = X_{M_\lambda}$ for some $\lambda \in \mathbb{P}_0^1$. Fix E a quasi-simple module in an exceptional tube \mathcal{T} , we know from [Dup09] that X_{M_E} is a polynomial in the X_{E_i} where the E_i are the quasi-simples of \mathcal{T} . In particular each X_{E_i} being a cluster variable, it follows that $X_{M_E} \in \mathcal{A}(Q)$. As $\text{q.rad} M_E/E$ is indecomposable rigid, $X_{\text{q.rad} M_E/E}$ is also a cluster variable and thus

$$X_\delta = X_{M_\lambda} = X_{M_E} - X_{\text{q.rad} M_E/E} \in \mathcal{A}(Q).$$

Now, we know that $X_{\mathbf{d}} \in \mathcal{A}(Q)$ for any Schur root \mathbf{d} . Fix now any element $\mathbf{d} \in \mathbb{N}^{Q_0}$, and write $\mathbf{d} = \mathbf{e}_1 \oplus \dots \oplus \mathbf{e}_n$ its canonical decomposition, then every \mathbf{e}_i is a Schur root and thus $X_{\mathbf{e}_i} \in \mathcal{A}(Q)$ for every $i = 1, \dots, n$. Now proposition 3.7 implies that

$$X_{\mathbf{d}} = \prod_{i=1}^n X_{\mathbf{e}_i} \in \mathcal{A}(Q),$$

this proves the claim.

Fix now an object M in \mathcal{C}_Q . According to proposition 2.14, X_M can be written as a \mathbb{Q} -linear combination of X_Y where $Y = \bigoplus_i Y_i$ is such that $\text{Ext}_{\mathcal{C}}^1(Y_i, Y_j) = 0$ for any $i \neq j$. It follows from lemma 3.34 that $X_Y \in \mathbb{Q}\mathcal{B}'(Q)$ and thus $X_M \in \mathbb{Q}\mathcal{B}'(Q)$.

Fix now a monomial $x \in \mathcal{A}(Q)$. Then $x = \prod_i x_i$ is a product of cluster variables x_i . According to theorem 1.3, each x_i can be written as a X_{M_i} and thus $x = X_{\bigoplus_i M_i} \in \mathbb{Q}\mathcal{B}'(Q)$. \square

4. REFLECTION FUNCTORS AND CALDERO-CHAPOTON MAP

4.1. Reflections and generic variables. We are now interested in proving that the elements in $\mathcal{B}'(Q)$ are linearly independent over \mathbb{Q} . For this purpose, we will follow the ideas of [CK08] introducing a certain grading on the cluster algebra. This grading will depend on the orientation of the quiver Q and thus we will need to understand the behaviour of generic variables under changes of orientations. In [Zhu06], the author investigated interactions between a certain extension of the BGP-reflection functors and the cluster combinatorics. This subsection is devoted to the study of the interaction between these extended reflection functors and the Caldero-Chapoton map. This is a generalization of the the works of Zhu.

4.1.1. Reflection functors and Caldero-Chapoton map. A *sink* (resp. *source*) in Q is a vertex in Q_0 such that there is no arrow starting (resp. ending) at i . Let Q be a quiver and i be a sink or a source in Q . We define the reflected quiver $\sigma_i(Q)$ by reversing all the arrows ending at i . An *admissible sequence of sinks* (resp. *sources*) is a sequence (i_1, \dots, i_n) such that i_1 is a sink (resp. source) in Q and i_k is a sink (resp source) in $\sigma_{i_{k-1}} \cdots \sigma_{i_1}(Q)$ for any $k = 2, \dots, n$. A quiver Q' is called *reflection-equivalent* to Q if there exists an admissible sequence of sinks or sources (i_1, \dots, i_n) such that $Q' = \sigma_{i_n} \cdots \sigma_{i_1}(Q)$. Note that this is an equivalence relation whose equivalence classes are called *reflection classes*. In the following, we will only work with sinks but a straightforward adaptation gives the same results for sources.

It is important to notice that mutations can be viewed as generalizations of reflections. Namely, if i is a sink or a source in a quiver Q , then $\mu_i(Q) = \sigma_i(Q)$ where μ_i denotes the mutation in the direction i .

If Q is a quiver, we still denote by $\mathcal{A}(Q)$ the coefficient free cluster algebra with initial seed (Q, \mathbf{u}) . If Q' is a quiver mutation-equivalent to Q , there exists some seed (Q', \mathbf{v}) in $\mathcal{A}(Q)$ and $\mathcal{A}(Q')$ will denote the cluster algebra with initial seed (Q', \mathbf{v}) . There is a natural isomorphism of cluster algebras

$$\Phi' : \mathcal{A}(Q') \longrightarrow \mathcal{A}(Q)$$

sending v_i to its expansion in $\mathbb{Z}[u_i^{\pm 1}, i \in Q_0]$ for any $i \in Q_0$. Similarly, every u_i can be written as a Laurent polynomial in \mathbf{v} and we write

$$\Phi : \mathcal{A}(Q) \longrightarrow \mathcal{A}(Q')$$

the corresponding algebra isomorphism. Note in particular that Φ' and Φ are inverse isomorphisms. These isomorphisms will be referred to as the *canonical cluster algebras isomorphisms*.

From now on, we assume that Q is acyclic and that i is a sink in Q_0 . We denote by $Q' = \sigma_i(Q)$ the reflected quiver. Let $\Sigma_i^+ : \text{rep}(Q) \longrightarrow \text{rep}(Q')$ be the standard BGP-reflection functor and $R_i^+ : \mathcal{C}_Q \longrightarrow \mathcal{C}_{Q'}$ be the extended BGP-reflection functor defined in [Zhu06]. It is given on the objects of \mathcal{C}_Q by:

$$R_i^+ : \begin{cases} X \mapsto \Sigma_i^+(X) & \text{if } X \not\cong S_i \text{ is a module} \\ S_i \mapsto P_i[1] \\ P_j[1] \mapsto P_j[1] \\ P_i[1] \mapsto S_i \end{cases} \quad \text{if } j \neq i$$

The following holds:

Proposition 4.1 ([Zhu06]). *Let Q be an acyclic quiver and i be a sink in \mathcal{C}_Q . Then:*

- (1) Σ_i^+ induces a triangle equivalence $D^b(kQ) \rightarrow D^b(kQ')$ commuting with the shift $[1]$ and the AR-translation τ .
- (2) R_i^+ induces a triangle equivalence $\mathcal{C}_Q \rightarrow \mathcal{C}_{Q'}$.

Set $T_i = \tau^{-1}P_i$, $T_j = P_j$ for $j \neq i$ and $T = \bigoplus_{k=1}^n T_k$ the APR-tilting module associated to the sink i and denote by the functor $F = \text{Hom}_{\mathcal{C}_Q}(T, -)$. It is known that F induces a triangle equivalence $\mathcal{C}_Q/\text{add } T[1] \rightarrow \text{mod-}k\sigma_i Q$. Moreover, F and R_i^+ coincide on the objects of $\mathcal{C}_Q/\text{add } T$.

We denote by X_{τ}^Q (resp. by $X_{\tau}^{\sigma_i Q}$) the Caldero-Chapoton map associated to Q (resp. to $\sigma_i Q$). We denote by X_{τ}^T the Palu's cluster character on \mathcal{C}_Q associated to the cluster-tilting object T introduced in [Pal08]. It is defined on indecomposable objects of \mathcal{C}_Q by

$$X_M^T = \begin{cases} \sum_{\mathbf{e}} \chi(\text{Gr}_{\mathbf{e}}(FM)) \prod_j v_j^{\langle S_j, \mathbf{e} \rangle_a - \langle S_j, FM \rangle} & \text{if } M \in \mathcal{C}_Q/\text{add } T[1] \\ v_j & \text{if } M \simeq T_j[1] \text{ for any } j \in Q_0 \end{cases}$$

where $\langle -, - \rangle_a$ is the symmetrized Euler form defined by

$$\langle M, N \rangle_a = \langle M, N \rangle - \langle N, M \rangle$$

for any two $k\sigma_i Q$ -modules M and N . In our case, as $\sigma_i Q$ is acyclic, the Euler form is thus well defined on the Grothendieck group of $k\sigma_i Q\text{-mod}$, thus

$$\langle S_j, \mathbf{e} \rangle_a = \langle S_j, \mathbf{e} \rangle - \langle \mathbf{e}, S_j \rangle$$

for any dimension vector \mathbf{e} and any $j \in \sigma_i Q_0$. It satisfies

$$X_{M \oplus N}^T = X_M^T X_N^T$$

for any two objects M, N in \mathcal{C}_Q . The following lemma gives the link between the cluster character and the Caldero-Chapoton map in the particular case where T is an APR-tilting module.

Lemma 4.2. *Let Q be an acyclic quiver, i a sink in Q and T be the APR-tilting kQ -module associated to i . Then for any object M in \mathcal{C}_Q , we have*

$$X_M^T = X_{R_i^+ M}^{\sigma_i Q}$$

Proof. As $X_{M \oplus N}^T = X_M^T X_N^T$ and $X_{M \oplus N}^{\sigma_i Q} = X_M^{\sigma_i Q} X_N^{\sigma_i Q}$, it suffices to prove the lemma for M indecomposable. If $M = T_i[1]$, then $R_i^+ M = P_i[1]$ and thus $X_{R_i^+ M}^{\sigma_i Q} = X_{P_i[1]}^{\sigma_i Q} = v_i$ but by definition $X_M^T = X_{T_i[1]}^T = v_i$. Let's assume that M is indecomposable and non-isomorphic to $T_i[1]$ for any $i \in Q_0$. As $R_i^+(M) = FM$, we compute

$$\begin{aligned} X_M^T &= \sum_{\mathbf{e}} \chi(\text{Gr}_{\mathbf{e}}(FM)) \prod_i v_i^{\langle S_i, \mathbf{e} \rangle_a - \langle S_i, FM \rangle} \\ &= \sum_{\mathbf{e}} \chi(\text{Gr}_{\mathbf{e}}(FM)) \prod_i v_i^{\langle S_i, \mathbf{e} \rangle - \langle \mathbf{e}, S_i \rangle - \langle S_i, FM \rangle} \\ &= \sum_{\mathbf{e}} \chi(\text{Gr}_{\mathbf{e}}(FM)) \prod_i v_i^{-\langle S_i, FM - \mathbf{e} \rangle - \langle \mathbf{e}, S_i \rangle} \\ &= X_{FM}^{\sigma_i Q} \\ &= X_{R_i^+ M}^{\sigma_i Q} \end{aligned}$$

□

4.1.2. *Reflections and cluster variables.* We keep the above notations. We denote by $\Phi_i : \mathcal{A}(Q) \rightarrow \mathcal{A}(\sigma_i Q)$ the canonical isomorphism of cluster algebras and by $\Phi'_i : \mathcal{A}(\sigma_i Q) \rightarrow \mathcal{A}(Q)$ the inverse isomorphism. It satisfies

$$\Phi_i(X_{T_j[1]}^Q) = X_{R_i^+ T_j[1]}^{\sigma_i Q} = X_{P_j[1]}^{\sigma_i Q} = v_j$$

for every $j \in Q_0$. As $T[1]$ is a cluster-tilting object in \mathcal{C}_Q , it follows from theorem 1.3 that $\{X_{T_j[1]}^Q, j \in Q_0\}$ is a cluster in $\mathcal{A}(Q)$ and $\{\Phi_i(X_{T_j[1]}^Q), j \in Q_0\} = \{X_{P_j[1]}^{\sigma_i Q}, j \in Q_0\}$ is a cluster in $\mathcal{A}(\sigma_i Q)$. We have the commutative diagram:

$$\begin{array}{ccc} \{T_j[1], j \in Q_0\} & \xrightarrow{R_i^+} & \{P_j[1], j \in Q_0\} \\ X_i^Q \downarrow & & \downarrow X_i^{\sigma_i Q} \\ \{X_{T_j[1]}, j \in Q_0\} & \xrightarrow{\Phi_i} & \{v_j, j \in Q_0\}. \end{array}$$

The following lemma is a consequence of the works of Zhu and of Palu. We give an independent proof for completeness.

Lemma 4.3. *Let Q be an acyclic quiver, i be a sink in \mathcal{C}_Q . Then*

$$\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$$

for every rigid object M in \mathcal{C}_Q .

Proof. Assume that R is a cluster-tilting object such that for every direct summand M of R , we have $\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$. Fix R' a cluster-tilting object in \mathcal{C}_Q next to R in the tilting graph (see [BMR⁺06] for terminology and results about cluster-tilting theory and exchange pairs). Then, there exists an exchange pair (U, U^*) such that $R = U \oplus \overline{U}$ and $R' = U^* \oplus \overline{U}$. We denote by

$$U \rightarrow B \rightarrow U^* \rightarrow U[1] \text{ and } U^* \rightarrow B' \rightarrow U \rightarrow U^*[1]$$

the corresponding triangles. Thus, B and B' are in $\text{add } \overline{U}$. According to theorem 2.7, we have

$$X_U^Q X_{U^*}^Q = X_B^Q + X_{B'}^Q$$

and thus

$$\Phi_i(X_U^Q) \Phi_i(X_{U^*}^Q) = \Phi_i(X_B^Q) + \Phi_i(X_{B'}^Q).$$

The induction gives

$$\Phi_i(X_{U^*}^Q) = \frac{X_{R_i^+ B}^Q + X_{R_i^+ B'}^Q}{X_{R_i^+ U}^Q}.$$

On the other hand R_i^+ is a triangle equivalence so $(R_i^+ U, R_i^+ U^*)$ is an exchange pair and the corresponding triangles are

$$R_i^+ U \rightarrow R_i^+ B \rightarrow R_i^+ U^* \rightarrow R_i^+ U[1] \text{ and } R_i^+ U^* \rightarrow R_i^+ B' \rightarrow R_i^+ U \rightarrow R_i^+ U^*[1]$$

The one-dimensional multiplication formula in $\mathcal{C}_{\sigma_i Q}$ gives:

$$X_{R_i^+ U^*}^{\sigma_i Q} = \frac{X_{R_i^+ B}^{\sigma_i Q} + X_{R_i^+ B'}^{\sigma_i Q}}{X_{R_i^+ U}^{\sigma_i Q}}$$

and thus

$$X_{R_i^+ U^*}^{\sigma_i Q} = \Phi_i(X_{U^*}^Q).$$

According to lemma 4.2, the proposition holds for the cluster-tilting object $T[1]$ and as the tilting graph is connected (see [BMR⁺06] and [HU05]), it follows that for every direct summand M of a tilting object, we have $\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$. As every indecomposable rigid object M can be completed into a cluster-tilting object, it follows that

$$\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$$

for every rigid object M in \mathcal{C}_Q . This proves the lemma. \square

4.1.3. Reflections for affine quivers. We now return to the case when Q is an affine quiver.

Lemma 4.4. *Let Q be an affine quiver and $i \in Q_0$ be a sink. The following hold:*

- (1) *A indecomposable kQ -module M is a regular module if and only if $\Sigma_i^+(M)$ is a regular $k\sigma_i Q$ -module.*
- (2) *Let M be an indecomposable regular kQ -module. Then Σ_i^+ induces an equivalence of categories from the tube containing M in $kQ\text{-mod}$ to the tube containing $\Sigma_i^+ M$ in $k\sigma_i Q\text{-mod}$.*

Proof. We recall that Σ_i^+ induces an equivalence $kQ\text{-mod}/S_i \rightarrow k\sigma_i Q\text{-mod}/S_i$. As the simple kQ -module S_i is projective and the simple $k\sigma_i Q$ -module S_i is an injective kQ -module. Denote by $\pi : kQ\text{-mod} \rightarrow kQ\text{-mod}/S_i$ the canonical functor. Then for any regular component \mathcal{T} , the restriction of the functor π to \mathcal{T} is isomorphic to the restriction of the identity to \mathcal{T} . If M is a regular module, then it belongs to a tube \mathcal{T} and $\Sigma_i^+ M \simeq \Sigma_i^+(\pi(M))$ belongs to $\Sigma_i^+(\pi(\mathcal{T}))$. But $\Sigma_i^+(\pi(\mathcal{T})) \simeq \pi(\mathcal{T}) \simeq \mathcal{T}$ and then $\Sigma_i^+(M)$ belongs to a tube in $k\sigma_i Q\text{-mod}/S_i$, it is thus a regular module and moreover Σ_i^+ induces an equivalence between the tubes containing M and $\Sigma_i^+(M)$. \square

Corollary 4.5. *Let Q be an affine quiver, i be a sink in Q_0 and M be an indecomposable regular kQ -module with quasi-composition series*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

then $R_i^+ M$ is a regular $k\sigma_i Q$ -module with regular composition series

$$0 = R_i^+ M_0 \subset R_i^+ M_1 \subset \cdots \subset R_i^+ M_r = R_i^+ M.$$

In particular R_i^+ send quasi-socles to quasi-socles, quasi-radicals to quasi-radicals and preserves quasi-lengths.

Proof. As i is a sink, S_i is a projective module and then for any regular module $R_i^+(M) \simeq \Sigma_i^+(M)$. The result follows then directly from lemma 4.4. \square

Now we extend lemma 4.3 to any object M in \mathcal{C}_Q .

Proposition 4.6. *Let Q be a quiver of affine type with at least three vertices satisfying the difference property. Let i be a sink in Q_0 such that $\sigma_i Q$ satisfies the difference property. Denote by $\Phi_i : \mathcal{A}(Q) \rightarrow \mathcal{A}(\sigma_i Q)$ the canonical isomorphism and by $R_i^+ : \mathcal{C}_Q \rightarrow \mathcal{C}_{\sigma_i Q}$ the extended BGP functor. Then for any object M in \mathcal{C}_Q , we have $\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$.*

Proof. It suffices to prove it for M indecomposable. If M is rigid the result follows from lemma 4.3. If M is not rigid then it is regular.

Assume first that M is in an exceptional tube \mathcal{T} of rank $p > 1$. Denote by E_0, \dots, E_{p-1} the quasi-simple modules of \mathcal{T} ordered such that $\tau E_i = E_{i-1}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. Denote by l the quasi-length of M and assume that the E_i are indexed in such a way that $E_0 = \text{q.soc } M$. Then we know from [Dup09] that

$$X_M^Q = P_{l,p}(X_{E_0}^Q, \dots, X_{E_{p-1}}^Q)$$

where $P_{l,p}$ is the l -th generalized Chebyshev polynomial of rank p .

According to lemma 4.4, $R_i^+ M$ is regular and belongs to a tube \mathcal{T}' of rank p . Moreover corollary 4.5 implies that the quasi-simples of \mathcal{T}' are the $R_i^+ E_i$ for $i \in \mathbb{Z}/p\mathbb{Z}$ and $R_i^+ E_0 = \text{q.soc} R_i^+ M$. Moreover, $\text{q.l}(R_i^+ M) = \text{q.l}(M) = l$ and thus

$$X_{R_i^+ M}^{\sigma_i Q} = P_{l,p}(X_{R_i^+ E_0}^{\sigma_i Q}, \dots, X_{R_i^+ E_{p-1}}^{\sigma_i Q}).$$

As every quasi-simple in an exceptional tube is rigid, it follows that

$$X_{R_i^+ E_i}^{\sigma_i Q} = \Phi_i(X_{E_i}^Q) \text{ for any } i \in \mathbb{Z}/p\mathbb{Z}$$

and thus

$$\begin{aligned} X_{R_i^+ M}^{\sigma_i Q} &= P_{l,p}(X_{R_i^+ E_0}^{\sigma_i Q}, \dots, X_{R_i^+ E_{p-1}}^{\sigma_i Q}) \\ &= P_{l,p}(\Phi_i(X_{E_0}^Q), \dots, \Phi_i(X_{E_{p-1}}^Q)) \\ &= \Phi_i(P_{l,p}(X_{E_0}^Q, \dots, X_{E_{p-1}}^Q)) \\ &= \Phi_i(X_M^Q) \end{aligned}$$

It only remains to prove the result for M indecomposable in an homogeneous tube. R_i^+ is a triangle equivalence $\mathcal{C}_Q \rightarrow \mathcal{C}_{\sigma_i Q}$ so according to lemma 4.4, R_i^+ send the homogeneous tubes of \mathcal{C}_Q to the homogeneous tubes of $\mathcal{C}_{\sigma_i Q}$. For any $\lambda \in \mathbb{P}^1(k)$, we denote by $\mathcal{T}_\lambda(Q)$ (resp. $\mathcal{T}_\lambda(\sigma_i Q)$) the tube in \mathcal{C}_Q (resp. $\mathcal{C}_{\sigma_i Q}$) corresponding to the parameter λ . Up to re-indexation of the tubes, we can assume that $R_i^+ \mathcal{T}_\lambda(Q) = \mathcal{T}_\lambda(\sigma_i Q)$ for any $\lambda \in \mathbb{P}^1(k)$ and lemma 4.4 implies that $\mathbb{P}_0^1(\sigma_i Q) = \mathbb{P}_0^1(Q)$. The set $\{M_\lambda\}_{\lambda \in \mathbb{P}_0^1}$ is a set of representatives of the quasi-simple modules in the homogeneous tubes in \mathcal{C}_Q and $\{R_i^+ M_\lambda\}_{\lambda \in \mathbb{P}_0^1}$ is a set of representatives of the quasi-simple modules in the homogeneous tubes in $\mathcal{C}_{\sigma_i Q}$. Moreover, it follows from lemma 3.14 and corollary 3.13 that

$$X_{M_\lambda}^{Q(n)} = X_{M_\mu}^{Q(n)} = C_n(X_{M_\lambda}^Q) \text{ and } X_{R_i^+ M_\lambda}^{\sigma_i Q(n)} = X_{R_i^+ M_\mu}^{\sigma_i Q(n)} = C_n(X_{R_i^+ M_\lambda}^{\sigma_i Q})$$

for any $\lambda, \mu \in \mathbb{P}_0^1(Q)$. It thus suffices to prove that

$$\Phi_i(X_{M_\lambda}^Q) = X_{R_i^+ M_\lambda}^{\sigma_i Q}$$

for some $\lambda \in \mathbb{P}_0^1(Q)$.

The difference property (4) of Q implies that for any quasi-simple E in an exceptional tube, we have

$$X_{M_\lambda}^Q = X_{M_E}^Q - X_{\text{q.rad} M_E / E}^Q$$

and the difference property (4) of $\sigma_i Q$

$$X_{R_i^+ M_\lambda}^{\sigma_i Q} = X_{R_i^+ M_E}^{\sigma_i Q} - X_{\text{q.rad} R_i^+ M_E / \text{q.soc} R_i^+ M_E}^{\sigma_i Q}$$

In particular,

$$\begin{aligned} \Phi_i(X_{M_\lambda}^Q) &= \Phi_i(X_{M_E}^Q) - \Phi_i(X_{\text{q.rad} M_E / E}^Q) \\ &= X_{R_i^+ (M_E)}^{\sigma_i Q} - X_{R_i^+ (\text{q.rad}(M_E / E))}^{\sigma_i Q} \end{aligned}$$

It follows from lemma 4.4 that $R_i^+(M_E) = M_{R_i^+(E)}$ and $\text{q.rad} R_i^+ M_E / \text{q.soc} R_i^+ M_E = \text{q.rad} M_{R_i^+(E)} / R_i^+(E)$ and thus

$$\begin{aligned} \Phi_i(X_{M_\lambda}^Q) &= X_{M_{R_i^+(E)}}^{\sigma_i Q} - X_{\text{q.rad} M_{R_i^+(E)} / R_i^+(E)}^{\sigma_i Q} \\ &= X_{R_i^+ M_\lambda}^{\sigma_i Q}. \end{aligned}$$

□

A direct induction on proposition 4.6 leads to the following proposition:

Proposition 4.7. *Let Q be a quiver of affine type with at least three vertices satisfying the difference property. Let Q' be a quiver reflection-equivalent to Q and (i_1, \dots, i_n) be an admissible sequence of sinks such that $Q' = \sigma_{i_n} \circ \dots \circ \sigma_{i_1}(Q)$. Assume moreover that $\sigma_{i_k} \dots \sigma_{i_1}(Q)$ satisfies the difference property for every $k = 1, \dots, n$. Denote by Φ the canonical isomorphism of cluster algebras $\mathcal{A}(Q) \rightarrow \mathcal{A}(Q')$ and by*

$$R^+ = R_{i_n}^+ \dots R_{i_1}^+ : \mathcal{C}_Q \rightarrow \mathcal{C}_{Q'}$$

Then

$$\Phi(X_M^Q) = X_{R^+M}^{Q'}$$

for every object M in \mathcal{C}_Q .

4.1.4. *Reflections and generic variables.* We are now able to prove that the generic variables are preserved under reflections.

Theorem 4.8. *Let Q be an affine quiver of affine type with at least three vertices such that every quiver reflection-equivalent to Q satisfies the difference property. Let Q' be a quiver reflection-equivalent to Q . Write $\Phi : \mathcal{A}(Q) \rightarrow \mathcal{A}(Q')$ the canonical isomorphism of cluster algebras. Then*

$$\Phi(\mathcal{B}'(Q)) = \mathcal{B}'(Q').$$

Proof. Let (i_1, \dots, i_n) be an admissible sequence of sinks such that $Q' = \sigma_{i_n} \circ \dots \circ \sigma_{i_1}(Q)$. We denote by R^+ the composition

$$R^+ = R_{i_n}^+ \dots R_{i_1}^+ : \mathcal{C}_Q \rightarrow \mathcal{C}_{Q'},$$

it is an equivalence of triangulated categories. According to proposition 4.7, we have $\Phi(X_M^Q) = X_{R^+M}^{Q'}$ for every object M in \mathcal{C}_Q .

Fix $\mathbf{d} \in \mathbb{Z}^{Q_0}$ and assume that $\mathbf{d} \notin \mathbb{N}^{Q_0}$. Then

$$X_{\mathbf{d}} = X_{[\mathbf{d}]_+} \prod_{d_i < 0} X_{P_i[1]^{\oplus(-d_i)}}^Q$$

Because $[\mathbf{d}]_+$ is not sincere, there exists some $M_+ \in \text{rep}(Q, [\mathbf{d}]_+)$ which is rigid. It follows that $M = M_+ \oplus \bigoplus_{d_i < 0} P_i[1]^{-d_i}$ is also rigid and thus so is R^+M . Then, $\Phi(X_M^Q) = X_{R^+M}^{Q'}$ is in $\mathcal{B}'(Q')$.

If $\mathbf{d} \in \mathbb{N}^{Q_0}$, assume that there is some rigid module $M \in U_{\mathbf{d}}$, then R^+M is rigid and thus $\Phi(X_M^Q) = X_{R^+M}^{Q'}$ is in $\mathcal{B}'(Q')$. Otherwise, according to the canonical decomposition of $[\mathbf{d}]_+$, $M \in U_{[\mathbf{d}]_+} \cap \mathfrak{M}_{[\mathbf{d}]_+}$ decomposes into

$$M = \bigoplus_{i \in I} M_{\lambda_i} \oplus \bigoplus_{j \in J} N_j$$

where the N_j are indecomposable rigid modules with trivial endomorphism ring for all $i, j \in J$ and $\text{Ext}_{\mathcal{C}_Q}^1(U, V) = 0$ for any pairwise distinct indecomposable summand of M and I is a non-empty set. It follows that

$$R^+M = \bigoplus_{i \in I} R^+M_{\lambda_i} \oplus \bigoplus_{j \in J} R^+N_j$$

where the R^+N_j are indecomposable rigid modules with local endomorphism ring for all $i, j \in J$ and $\text{Ext}_{\mathcal{C}_Q}^1(U, V) = 0$ for any pairwise distinct indecomposable summand of R^+M . Assume that $R^+N_j \simeq P_k[1]$ for some projective kQ' -module P_k . Then

$$\text{Ext}_{\mathcal{C}_{Q'}}^1(R^+N_j, M_{\lambda_i}) = \text{Ext}_{\mathcal{C}_{Q'}}^1(P_k[1], M_{\lambda_i}) = \dim M_{\lambda_i}(k) > 0$$

which is a contradiction. It follows that R^+M is a kQ' -module and according to the study in subsection 3.1.2, we have

$$X_{R^+M}^{Q'} = X_{\mathbf{dim} R^+M}^{Q'}$$

and thus $\Phi(\mathcal{B}'(Q)) \subset \mathcal{B}'(Q')$.

Conversely, Q can be obtained from Q' after a sequence of reflections and thus the same proof shows the inverse inclusion. \square

If $i \in Q_0$, we denote by s_i the standard reflection of the Weyl group associated to i . We denote by σ_i the piecewise linear transformation defined on $\mathbb{Z}\Phi(Q)$ as follows. If $\alpha \in \Phi_{\geq -1}(Q)$, following [MRZ03] we set

$$\sigma_i(\mathbf{d}) = \begin{cases} \mathbf{d} & \text{if } \mathbf{d} = -\alpha_j \text{ for } j \neq i, \\ s_i(\mathbf{d}) & \text{otherwise.} \end{cases}$$

If $\mathbf{d} \in \mathbb{N}^{Q_0}$, we write $\mathbf{d} = \mathbf{e}_1 \oplus \cdots \oplus \mathbf{e}_n$ the canonical decomposition of \mathbf{d} and we set

$$\sigma_i(\mathbf{d}) = \sum_{j=1}^n \sigma_i(\mathbf{e}_j).$$

If $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we set

$$\sigma_i(\mathbf{d}) = \sigma_i([\mathbf{d}]_+) + [\mathbf{d}]_- + 2d_i\alpha_i.$$

This is an involution of \mathbb{Z}^{Q_0} .

Lemma 4.9. *Let Q be an acyclic quiver, i and sink in Q . Fix M an object in \mathcal{C}_Q such that*

$$M \simeq M_1 \oplus \cdots \oplus M_n$$

where the M_k are indecomposable objects such that $\text{Ext}_{\mathcal{C}_Q}^1(M_k, M_j) = 0$ for $k \neq j$. Then

$$\mathbf{dim} R_i^+(M) = \sigma_i(\mathbf{dim} M).$$

Proof. If $n = 1$, then $M = M_1$ is indecomposable and $\mathbf{dim} M \in \Phi_{\geq 1}$ and the result is proved in [Zhu06, theorem 3.4]. Assume now $n > 1$. As $\text{Ext}_{\mathcal{C}_Q}^1(M_k, M_j) = 0$ for $k \neq j$, if some M_j is isomorphic to some $P_k[1]$, then $(\mathbf{dim} M_l)_k \leq 0$ for any $l = 1, \dots, n$. It follows that

$$[\mathbf{dim} M]_+ = \mathbf{dim} H^0(M).$$

We write

$$M = H^0(M) \oplus \bigoplus_{k \in Q_0} P_k[1]^{\oplus p_k}.$$

In particular, for any $k \in Q_0$, we have $-p_k = (\mathbf{dim} M)_k$. We have

$$R_i^+(M) = \Sigma_i^+(H^0(M)) \oplus S_i^{\oplus p_i} \oplus \bigoplus_{k \neq i} P_k[1]^{\oplus p_k}$$

where Σ_i^+ denotes the standard BGP-reflection functor. It follows that

$$\begin{aligned} \mathbf{dim} R_i^+M &= s_i(\mathbf{dim} H^0(M)) + p_i\alpha_i - p_k\alpha_k \\ &= s_i([\mathbf{dim} M]_+) + [\mathbf{dim} M]_- + 2p_i\alpha_i \\ &= \sigma_i([\mathbf{dim} M]_+) + [\mathbf{dim} M]_- + 2p_i\alpha_i \\ &= \sigma_i(\mathbf{dim} M). \end{aligned}$$

\square

Corollary 4.10. *Let Q be an affine quiver of affine type with at least three vertices satisfying the difference property. Let i be a sink in Q such that $\sigma_i Q$ satisfies the difference property, and $\Phi_i : \mathcal{A}(Q) \rightarrow \mathcal{A}(\sigma_i Q)$ be the canonical isomorphism of cluster algebras. Then for any $\mathbf{d} \in \mathbb{Z}^{Q_0}$, we have*

$$\Phi_i(X_{\mathbf{d}}^Q) = X_{\sigma_i(\mathbf{d})}^{\sigma_i Q}.$$

Proof. According to the denominators theorem for any object M in \mathcal{C}_Q , we have

$$\dim R_i^+(M) = \delta(X_{R_i^+ M}^{\sigma_i Q}).$$

Fix $\mathbf{d} \in \mathbb{Z}^{Q_0}$ and M such that $X_M = X_{\mathbf{d}}$. Then we can choose M such that it satisfies the hypothesis of lemma 4.9, in this case, we have $\dim R_i^+(M) = \sigma_i(\dim M)$. Now, we know from proposition 4.6 that $\Phi_i(X_M^Q) = X_{R_i^+ M}^{\sigma_i Q}$ and thus it follows from theorem 4.8 that

$$\Phi_i(X_{\mathbf{d}}^Q) = X_{\sigma_i(\mathbf{d})}^{\sigma_i Q}.$$

□

4.2. Linear independence for generic values.

4.2.1. Gradability and linear independence. In [CK08], a condition was introduced on the quiver Q in order to obtain a nice framework for problems of linear independence of generalized variables. In the sequel, we will refer to this condition as *gradability*. For results concerning gradability, one can also refer to [CK08, Dup09].

We recall that for any acyclic quiver Q , the *matrix B associated to Q* is the anti-symmetric matrix whose entries are given by

$$b_{ij} = |\{i \rightarrow j \in Q_1\}| - |\{j \rightarrow i \in Q_1\}|$$

for all $i, j \in Q_0$.

Definition 4.11. Let Q be an acyclic quiver with associated matrix B . Q will be called *graded* if there exists a linear form ϵ on \mathbb{Z}^{Q_0} such that $\epsilon(B\alpha_i) < 0$ for any $i \in Q_0$ where α_i still denotes the i -th vector of the canonical basis of \mathbb{Z}^{Q_0} .

An orientation of Q is called *alternating* if every vertex in Q_0 is either a sink or a source.

Example 4.12. Every quiver equipped with an alternating orientation is graded. Indeed, in this case the matrix B of Q satisfies for all $j \in Q_0$, $b_{ij} > 0$ if i is a source and $b_{ij} < 0$ if i is a sink. We can choose any form ϵ in the dual basis of \mathbb{Z}^{Q_0} such that $\epsilon_i < 0$ if i is a source and $\epsilon_i > 0$ if i is a sink.

If Q is a graded quiver, then it is proved in [CK08] that we can endow $\mathcal{A}(Q)$ with a grading. Namely, the results are the following:

For any Laurent polynomial L in the variables \mathbf{u} , the *support* $\text{supp}(L)$ of L is defined as the set of points $\lambda = (\lambda_i, i \in Q_0)$ of \mathbb{Z}^{Q_0} such that the λ -component, that is, the coefficient of $\prod_{i \in Q_0} u_i^{\lambda_i}$ in L is non-zero. For any λ in \mathbb{Z}^{Q_0} , let C_λ be the convex cone with vertex λ and edge vectors generated by the $B\alpha_i$ for any $i \in Q_0$.

Proposition 4.13 ([CK08]). *Let Q be an acyclic quiver with no multiple arrows. Fix an indecomposable object M of \mathcal{C}_Q and write $M = H^0(M) \oplus P_M[1]$. Then, $\text{supp}(X_M)$ is in C_{λ_M} with $\lambda_M := (-\langle \alpha_i, \dim H^0(M) \rangle + \langle \dim P_M, \alpha_i \rangle)_{i \in Q_0}$. Moreover, the λ_M -component of X_M is 1.*

Proposition 4.14 ([CK08]). *Let Q be a graded acyclic quiver with no multiple arrows. For every $n \in \mathbb{Z}$, set*

$$F_n = \left(\bigoplus_{\epsilon(\nu) \leq n} \mathbb{Z} \prod_{i \in Q_0} u_i^{\nu_i} \right) \cap \mathcal{A}(Q),$$

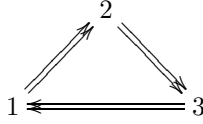
then the set $(F_n)_{n \in \mathbb{Z}}$ defines a \mathbb{Z} -filtration of $\mathcal{A}(Q)$.

The filtration given in proposition 4.14 and the proposition 4.13 imply the following lemma:

Lemma 4.15 ([CK08]). *Fix Q a graded quiver. Let $\{M_1, \dots, M_r\}$ be a family of objects in \mathcal{C}_Q such that $\mathbf{dim} M_i \neq \mathbf{dim} M_j$ if $i \neq j$, then $X_{M_1}^Q, \dots, X_{M_r}^Q$ are linearly independent over \mathbb{Q} .*

Definition 4.16. A quiver Q is called *gradable* if there exists a graded quiver Q' reflection-equivalent to Q .

Remark 4.17. Note that not all the quivers are gradable. For example, consider the 3-cycle Q with double arrows:



Denote by c_i the i -th column of the matrix B associated to Q . Then $c_3 = -c_1 - c_2$ and Q is the only quiver in its reflection class (and also in its mutation class, see [FZ03] for definitions).

We will see in the next subsection that all the quivers of affine types \tilde{A} are gradable.

Proposition 4.18. *Let Q be a gradable quiver of affine type \tilde{A} such that every quiver reflection-equivalent to Q satisfies the difference property. Let $\{M_1, \dots, M_r\}$ be a family of objects such that $\mathbf{dim} M_i \neq \mathbf{dim} M_j$ if $i \neq j$. Then $\{X_{M_1}^Q, \dots, X_{M_r}^Q\}$ is linearly independent over \mathbb{Q} .*

Proof. If Q is the Kronecker quiver, then every quiver Q' reflection-equivalent to Q is graded and thus the proposition holds by lemma 4.15. From now on, we assume that Q has at least three vertices. Fix Q' a graded quiver reflection-equivalent to Q and denote by (i_1, \dots, i_n) an admissible sequence of sinks such that $Q' = \sigma_{i_n} \circ \dots \circ \sigma_{i_1}(Q)$. We denote by

$$R^+ = R_{i_n}^+ \circ \dots \circ R_{i_1}^+ : \mathcal{C}_Q \longrightarrow \mathcal{C}_{Q'}$$

the equivalence of triangulated categories from \mathcal{C}_Q to $\mathcal{C}_{Q'}$ and by σ the composition of piecewise linear transformations

$$\sigma_{i_n} \circ \dots \circ \sigma_{i_1} : \mathbb{Z}\Phi(Q) \longrightarrow \mathbb{Z}\Phi(Q').$$

We know in particular that for any objects M, N such that $\mathbf{dim} M \neq \mathbf{dim} N$, we have $\mathbf{dim} R^+ M = \sigma \mathbf{dim} M \neq \sigma \mathbf{dim} N = \mathbf{dim} R^+ N$.

If we denote by $\Phi : \mathcal{A}(Q) \longrightarrow \mathcal{A}(Q')$ the canonical isomorphism, it follows from proposition 4.7 that

$$\Phi(X_M^Q) = X_{R^+ M}^{Q'}$$

for any object M in \mathcal{C}_Q . In particular

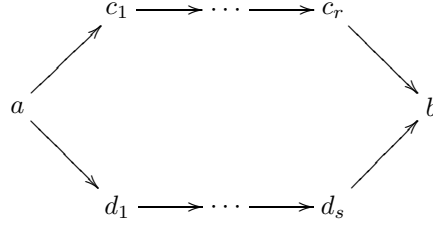
$$\{\Phi(X_{M_1}^Q), \dots, \Phi(X_{M_r}^Q)\} = \{X_{R^+ M_1}^{Q'}, \dots, X_{R^+ M_r}^{Q'}\}$$

satisfies the conditions of lemma 4.15. Q' being graded it follows that the family $\{\Phi(X_{M_1}^Q), \dots, \Phi(X_{M_r}^Q)\}$ is linearly independent over \mathbb{Q} . As Φ is a \mathbb{Q} -algebra homomorphism, it follows that $\{X_{M_1}^Q, \dots, X_{M_r}^Q\}$ is linearly independent over \mathbb{Q} . \square

4.2.2. Gradability for quivers of affine type $\tilde{\mathbb{A}}$. As claimed before, we now prove that any quiver of affine type $\tilde{\mathbb{A}}$ is gradable. This will be done by a case-by-case inspection depending on the indices r, s of the type $\tilde{\mathbb{A}}_{r,s}$.

Lemma 4.19. *Let Q be a quiver of affine type $\tilde{\mathbb{A}}$. Then Q is gradable.*

Proof. We fix $r, s \in \mathbb{N}$ two integers and we consider a quiver Q of affine type $\tilde{\mathbb{A}}_{r+1,s+1}$. We recall that all the quivers of affine type $\tilde{\mathbb{A}}_{r+1,s+1}$ are reflection-equivalent. We will thus always assume that Q is equipped with the following orientation:

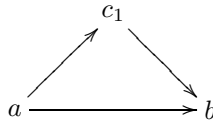


We will do a case-by-case inspection on the possible pairs (r, s) . By symmetry, it is sufficient to consider the pairs $\{r, s\}$. In each case, we will construct a form ϵ such that Q satisfies the condition of gradability for this ϵ . If ϵ is such a form, we will simply say that ϵ fits to Q .

If $B \in M_{Q_0}(\mathbb{Z})$ is a matrix, we denote by $C_i = B\alpha_i$ the column associated to the vertex $i \in Q_0$.

Assume first that $s = r = 0$. Then Q is the Kronecker quiver $a \rightrightarrows b$, it has an alternating orientation and thus by example 4.12, such a form exists.

Assume now that $\{r, s\} = \{0, 1\}$. We assume that $r = 1$ and $s = 0$. Then Q is the quiver

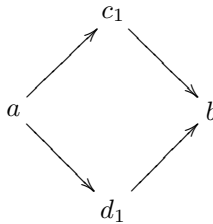


the associated matrix with indexation (a, c_1, b) is

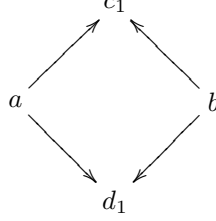
$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

The columns C_a and C_{c_1} are linearly independent and $C_b = C_{c_1} - C_a$. We set $\epsilon(C_a) = -1$, $\epsilon(C_{c_1}) = -2$ and then $\epsilon(C_b) = -1 < 0$. Thus ϵ fits.

Assume that $r = s = 1$. Then Q is the quiver

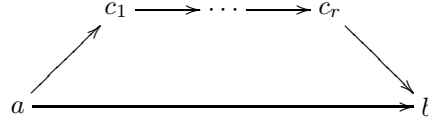


In the associated matrix B , we have $C_b = -C_a$, thus we will not be able to find any form ϵ fitting to Q . Consider then the quiver $Q' = \mu_b(Q) = \sigma_b(Q)$ given by



Q' is alternating, so it is graded and Q is gradable.

Now assume that $s = 0$ and $r \geq 2$ (or equivalently $r = 0$ and $s \geq 2$). Then Q is the quiver



The associated matrix in the indexation (a, c_1, \dots, c_r, b) is thus

$$\left[\begin{array}{c|ccc|cc} 0 & 1 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 1 & & & \\ \hline & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & -1 & 0 & 1 \\ \hline -1 & & & & -1 & 0 \end{array} \right]$$

and we have

$$\sum \lambda_j C_j = \begin{bmatrix} \lambda_b + \lambda_{c_1} \\ \lambda_{c_2} - \lambda_a \\ \vdots \\ \lambda_b - \lambda_{c_{r-1}} \\ -\lambda_{c_r} - \lambda_a \end{bmatrix}$$

If $r \in 2\mathbb{Z}$, then

$$\sum \lambda_j C_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = -\lambda_b \\ \lambda_a = \lambda_{c_2} = \cdots = \lambda_{c_r} \\ \lambda_b = \lambda_{c_{r-1}} = \cdots = \lambda_{c_1} \\ \lambda_a = -\lambda_{c_r} \end{cases} \Leftrightarrow \lambda_j = 0 \text{ for all } j \in Q_0$$

Thus, B is of full rank and Q is graded.

If $r \in 2\mathbb{Z} + 1$, then

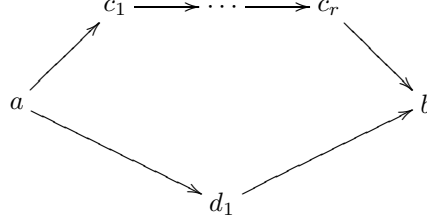
$$\sum_{j \neq b} \lambda_j C_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = 0 \\ \lambda_a = \lambda_{c_2} = \cdots = \lambda_{c_{r-1}} \\ \lambda_{c_{r-1}} = \lambda_{c_{r-3}} = \cdots = \lambda_{c_2} = 0 \\ \lambda_{c_r} = \cdots = \lambda_{c_{r-2}} = \cdots = \lambda_{c_1} \\ \lambda_a = -\lambda_{c_r} \end{cases} \Leftrightarrow \lambda_j = 0 \text{ for all } j$$

Thus, the columns $(C_a, C_{c_1}, \dots, C_{c_r})$ are linearly independent and

$$C_b = -C_a + C_{c_1} - C_{c_2} + C_{c_3} - \cdots - C_{c_{r-1}} + C_{c_r}$$

We thus set, $\epsilon(C_i) = -1$ for all $i \neq c_1, b$ and $\epsilon(C_{c_1}) = -2$. Then $\epsilon(C_b) < 0$ and ϵ fits to the condition.

Assume now that $s = 1$ and $r \geq 2$ (or equivalently $r = 1$ and $s \geq 2\mathbb{Z}$) Then Q is the quiver



The associated matrix in the indexation $(a, c_1, \dots, c_r, b, d_1)$ is

$$\left[\begin{array}{c|ccc|c|c} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \hline -1 & 0 & 1 & & & & \\ 0 & -1 & 0 & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & -1 & 0 & 1 & \\ \hline & & & & -1 & 0 & -1 \\ \hline -1 & & & & & 1 & 0 \end{array} \right].$$

We have

$$\sum_{j \in Q_0} \lambda_j C_j = \begin{bmatrix} \lambda_{c_1} + \lambda_{d_1} \\ \lambda_{c_2} - \lambda_a \\ \lambda_{c_3} - \lambda_{c_1} \\ \vdots \\ \lambda_b - \lambda_{c_{r-1}} \\ -\lambda_{d_1} - \lambda_{c_r} \\ \lambda_b - \lambda_a \end{bmatrix}$$

If $r \in 2\mathbb{Z}$,

$$\sum_{j \in Q_0} \lambda_j C_j = 0 \Leftrightarrow \lambda_a = \lambda_b = \lambda_{c_1} = \lambda_{c_2} = \cdots = \lambda_{c_r} = -\lambda_{d_1}$$

So $\sum_{j \neq d_1} \lambda_j C_j = 0 \Rightarrow \lambda_j = 0$ for all j and thus the columns $(C_a, C_{c_1}, \dots, C_{c_r}, C_b)$ are linearly independent and C_{d_1} is in the spanning of the other columns. More precisely,

$$\sum_{j \neq d_1} \lambda_j C_j = \begin{bmatrix} \lambda_{c_1} \\ \lambda_{c_2} - \lambda_a \\ \lambda_{c_3} - \lambda_{c_1} \\ \vdots \\ \lambda_b - \lambda_{c_{r-1}} \\ -\lambda_{c_r} \\ \lambda_b - \lambda_a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \Leftrightarrow \{ \lambda_{c_1} = \lambda_{c_2} = \cdots = \lambda_{c_r} = \lambda_b = \lambda_a = 1$$

Thus, $C_{d_1} = \sum_{j \neq d_1} C_j$ and thus it suffices to set $\epsilon(C_j) = -1$ for every $j \neq d_1$ and Q is graded.

If $r \in 2\mathbb{Z} + 1$, Q is not graded. Consider the quiver

$$Q' = \mu_b(Q) = \begin{array}{c} \begin{array}{ccccc} & c_1 & \longrightarrow & \cdots & \longrightarrow & c_r \\ & \nearrow & & & & \nwarrow \\ a & & & & & b \\ & \searrow & & & & \nearrow \\ & d_1 & & & & \end{array} \end{array}$$

The associated matrix in the indexation $(a, c_1, \dots, c_r, b, d_1)$ is

$$\left[\begin{array}{c|ccc|cc} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \hline -1 & 0 & 1 & & & & \\ 0 & -1 & 0 & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & -1 & 0 & -1 & \\ \hline & & & & 1 & 0 & 1 \\ \hline -1 & & & & & -1 & 0 \end{array} \right]$$

we have

$$\sum_{j \neq d_1} \lambda_j C_j = \begin{bmatrix} \lambda_{c_1} + \lambda_{d_1} \\ \lambda_{c_2} - \lambda_a \\ \vdots \\ \lambda_{c_r} - \lambda_{c_{r-2}} \\ -\lambda_b - \lambda_{c_{r-1}} \\ \hline \lambda_{c_r} + \lambda_{d_1} \\ \hline -\lambda_b - \lambda_a \end{bmatrix}.$$

Then,

$$\sum_{j \in Q_0} \lambda_j C_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = \lambda_{c_3} = \cdots = \lambda_{c_r} = -\lambda_{d_1} \\ \lambda_a = \lambda_{c_2} = \cdots = \lambda_{c_{r-1}} = -\lambda_b \end{cases}$$

So the columns $(C_a, C_{c_1}, \dots, C_{c_r})$ are linearly independent and C_b and C_{d_1} are in their spanning. More precisely, we have

$$C_b = C_a + C_{c_2} + \cdots + C_{c_{r-1}} \text{ and } C_{d_1} = C_{c_1} + C_{c_3} + \cdots + C_{c_r}$$

We thus set $\epsilon(C_i) = -1$ for all $i \neq b, d_1$ and thus ϵ fits. It follows that Q' is graded and Q is gradable.

Now we assume that $r, s \geq 2$, Q is the quiver

$$\begin{array}{ccccc} & c_1 & \longrightarrow & \cdots & \longrightarrow & c_r \\ & \nearrow & & & & \searrow \\ a & & & & & b \\ & \searrow & & & & \nearrow \\ & d_1 & \longrightarrow & \cdots & \longrightarrow & d_s \end{array}$$

the associated matrix B in the indexation $(a, c_1, \dots, c_r, b, d_1, \dots, d_s)$ is

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & & & & & & & \\ 0 & -1 & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & 1 & & & & & \\ & & & -1 & 0 & 1 & & & & \\ \hline & & & & -1 & 0 & & & & -1 \\ -1 & & & & & & 0 & 1 & & \\ & & & & & & -1 & \ddots & \ddots & \\ & & & & & & & \ddots & \ddots & 1 \\ & & & & & & & & -1 & 0 \\ & & & & & & 1 & & & \end{array} \right]$$

We have

$$\sum_{j \in Q_0} \lambda_j C_j = \begin{bmatrix} \lambda_{c_1} + \lambda_{d_1} \\ \lambda_{c_2} - \lambda_a \\ \lambda_{c_3} - \lambda_{c_1} \\ \vdots \\ \lambda_{c_r} - \lambda_{c_{r-2}} \\ \lambda_b - \lambda_{c_{r-1}} \\ -\lambda_{d_s} - \lambda_{c_r} \\ \lambda_{d_2} - \lambda_a \\ \lambda_{d_3} - \lambda_{d_1} \\ \vdots \\ \lambda_{d_s} - \lambda_{d_{s-2}} \\ \lambda_b - \lambda_{d_{s-1}} \end{bmatrix}$$

If $r, s \in 2\mathbb{Z}$, then

$$\sum_{j \in Q_0} \lambda_j C_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = -\lambda_{d_1} \\ \lambda_a = \lambda_{c_2} = \dots = \lambda_{c_r} \\ \lambda_b = \lambda_{c_{r-1}} = \dots = \lambda_{c_1} \\ \lambda_{c_r} = -\lambda_{d_s} \\ \lambda_a = \lambda_{d_2} = \dots = \lambda_{d_s} \\ \lambda_b = \lambda_{d_{s-1}} = \dots = \lambda_{d_1} \end{cases} \Leftrightarrow \lambda_i = 0 \text{ pour tout } i$$

and the columns are linearly independent. B is thus of full rank and Q is graded.

If $r \in 2\mathbb{Z}$ and $s \in 2\mathbb{Z} + 1$ (or equivalently $r \in 2\mathbb{Z} + 1$ and $s \in 2\mathbb{Z}$), then

$$\sum_{j \in Q_0} \lambda_j C_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = -\lambda_{d_1} \\ \lambda_a = \lambda_{c_2} = \dots = \lambda_{c_{r-1}} \\ \lambda_b = \lambda_{c_{r-1}} = \dots = \lambda_{c_1} \\ \lambda_{c_r} = -\lambda_{d_s} \\ \lambda_a = \lambda_{d_2} = \dots = \lambda_{d_s} = \lambda_b \\ \lambda_b = \lambda_{d_{s-1}} = \dots = \lambda_{d_1} \end{cases}$$

Thus, $\sum_{j \neq d_s} \lambda_j C_j = 0$ if and only if $\lambda_j = 0$ for all j . The columns $C_a, C_{c_1}, \dots, C_{c_r}, C_b, C_{d_1}, \dots, C_{d_{s-1}}$ are thus linearly independent and

$$C_{d_s} = C_a + C_{c_2} + \dots + C_{c_r} + C_{d_2} + \dots + C_{d_{s-1}} - C_{d_1} - \dots - C_{d_{s-2}}$$

We set $\epsilon(C_i) = -1$ for every $i \neq a, d_s$ and $\epsilon(C_a) = \min(0, r-1) - 1$. We have then $\epsilon(C_i) < 0$ for every $i \in Q_0$ and Q is graded.

If $r, s \in 2\mathbb{Z} + 1$, Q is not graded. We consider the quiver $Q' = \mu_b(Q)$ whose associated matrix in the indexation $(a, c_1, \dots, c_r, b, d_1, \dots, d_s)$ is

$$B' = \left[\begin{array}{c|cccc|c|cccc} 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & & & & & & & \\ 0 & -1 & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & 1 & & & & & \\ & & & -1 & 0 & -1 & & & & \\ \hline & & & & 1 & 0 & & & 1 & \\ -1 & & & & & & 0 & 1 & & \\ \hline & & & & & & -1 & \ddots & \ddots & \\ & & & & & & & \ddots & \ddots & 1 \\ & & & & & & & & -1 & 0 \end{array} \right].$$

If C'_i denotes the i -th column of B' for any i , we have

$$\sum_{j \in Q_0} \lambda_j C'_j = \begin{bmatrix} \lambda_{c_1} + \lambda_{d_1} \\ \lambda_{c_2} - \lambda_a \\ \lambda_{c_3} - \lambda_{c_1} \\ \vdots \\ \lambda_{c_r} - \lambda_{c_{r-2}} \\ -\lambda_b - \lambda_{c_{r-1}} \\ -\lambda_{d_s} - \lambda_{c_r} \\ \lambda_{d_2} - \lambda_a \\ \lambda_{d_3} - \lambda_{d_1} \\ \vdots \\ \lambda_{d_s} - \lambda_{d_{s-2}} \\ -\lambda_b - \lambda_{d_{s-1}} \end{bmatrix}$$

and

$$\sum \lambda_j C'_j = 0 \Leftrightarrow \begin{cases} \lambda_{c_1} = -\lambda_{d_1} \\ \lambda_a = \lambda_{c_2} = \cdots = \lambda_{c_{r-1}} = -\lambda_b \\ \lambda_{c_1} = \cdots = \lambda_{c_r} \\ \lambda_{c_r} = -\lambda_{d_s} \\ \lambda_a = \lambda_{d_2} = \cdots = \lambda_{d_{s-1}} = -\lambda_b \\ \lambda_{d_1} = \cdots = \lambda_{d_r} \end{cases}$$

Thus, $\sum_{j \neq b, d_s} \lambda_j C'_j = 0 \Leftrightarrow \lambda_j = 0$ for all j . The columns $C'_a, C'_{c_1}, \dots, C'_{c_r}, C'_{d_1}, \dots, C'_{d_{s-1}}$ are thus linearly independent. Moreover, we have

$$C'_b = C'_a + C'_{c_2} + \cdots + C'_{c_{r-1}} + C'_{d_2} + \cdots + C'_{d_{s-1}}$$

and

$$C'_{d_s} = C'_{c_1} + C'_{c_3} + \cdots + C'_{c_r} - C'_{d_1} - \cdots - C'_{d_{s-2}}$$

We thus set $\epsilon(C'_i) = -1$ for all $i \neq b, d_s, c_1$ and $\epsilon(C'_{c_1}) = \min(0, \frac{r+1}{2} - \frac{s-1}{2}) - 1$. Then, $\epsilon(C'_b) < 0$ and $\epsilon(C'_{d_s}) < 0$. Q' is thus graded and so Q is gradable.

In every case, we proved that a quiver Q of affine type $\tilde{A}_{r,s}$ is gradable. \square

Theorem 4.20. *Let Q be an affine quiver such that every quiver reflection-equivalent to Q satisfies the difference property. Then the set of generic variables is a \mathbb{Z} -basis in $\mathcal{A}(Q)$.*

Proof. Let Q be an affine quiver. If Q is of affine type \tilde{A} , then Q is gradable by lemma 4.19. Otherwise, the underlying diagram Δ of Q is a tree and thus admits an alternating orientation. As any orientations of a tree are reflection-equivalent, it follows that Q is gradable. By theorem 4.8, we can thus assume that Q is a graded quiver. It follows from lemma 4.15 that $\mathcal{B}'(Q)$ is linearly independent over \mathbb{Q} and thus over \mathbb{Z} .

Now, it follows from proposition 3.35 that every element X_M is a \mathbb{Q} -linear combination of elements of $\mathcal{B}'(Q)$. For every $\mathbf{d} \in \mathbb{Z}^{Q_0}$, set

$$\lambda_{\mathbf{d}} = (-\langle \alpha_i, [\mathbf{d}]_+ \rangle + \langle [\mathbf{d}]_-, \alpha_i \rangle)_{i \in Q_0},$$

If M is such that $X_M = X_{\mathbf{d}}$, then it follows from the definition of $X_{\mathbf{d}}$ that $\dim H^0(M) = [\mathbf{d}]_+$ and $\dim P_M[1] = [\mathbf{d}]_-$. Then, it follows from proposition 4.13 that $\text{supp}(X_{\mathbf{d}}) \in C_{\lambda_{\mathbf{d}}}$ and the $\lambda_{\mathbf{d}}$ component of $X_{\mathbf{d}}$ is 1.

As Q is graded, using the filtration given by proposition 4.14, we obtain

$$X_M = X_{\dim M} + \sum_{\epsilon(\mathbf{d}) < \epsilon(\dim M)} n_{\mathbf{d}} X_{\mathbf{d}}$$

where the $n_{\mathbf{d}}$ are rational numbers. By induction on the filtration, using the fact that $\chi(\text{Gr}_{\mathbf{e}}(M)) \in \mathbb{Z}$ for every dimension vector \mathbf{e} , we see that $n_{\mathbf{d}} \in \mathbb{Z}$ for every $\mathbf{d} \in \mathbb{Z}^{Q_0}$ and thus $\mathcal{A}(Q)$ is generated by $\mathcal{B}'(Q)$ as a \mathbb{Z} -module. \square

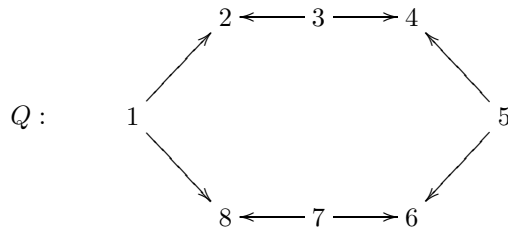
Theorem 4.21. *Let Q be a quiver of affine type \tilde{A} . Then $\mathcal{B}'(Q)$ is a \mathbb{Z} -basis of the \mathbb{Z} -module $\mathcal{A}(Q)$.*

Proof. According to theorem 3.25, a quiver of affine type \tilde{A} satisfies the difference property. Thus $\mathcal{B}'(Q)$ is a \mathbb{Z} -basis in $\mathcal{A}(Q)$ by theorem 4.20. \square

Remark 4.22. In the works of [CZ06] on the Kronecker quiver (see also [Cer08] for affine quivers of rank three), the elements arising in their semicanonical basis with denominator vectors $n\delta$ turn out to be the $X_{M_{\lambda}^{(n)}}$ for $n \geq 2$ and $\lambda \in \mathbb{P}_0^1$. Nevertheless, it follows from lemma 3.15 that $X_{n\delta} \neq X_{M_{\lambda}^{(n)}}$ and thus that generalized variables associated to the regular modules $M_{\lambda}^{(n)}$ for $n \geq 2$ do not appear as elements of $\mathcal{B}'(Q)$. Even if it seems more natural from the point of view of the AR quiver of \mathcal{C}_Q to introduce the $X_{M_{\lambda}^{(n)}}$, the natural choice from the point of view of geometry is $X_{M_{\lambda}^n}$. We will see in subsection 5.2 that for the Kronecker quiver, Caldero-Zelevinsky's semicanonical basis differ from the set $\mathcal{B}'(Q)$ of generic variables only by a locally unipotent (see subsection 5.2 for definitions) transformation.

The reader can find a bit confusing to use also the terminology *semicanonical* in our case whereas the considered set does not coincide with the semicanonical basis found in the previous literature. Nevertheless, because of the very strong analogy between our basis and Lusztig's dual semicanonical basis, we preferred to keep the terminology *semicanonical* even if it can seem a bit confusing at first.

Example 4.23. Consider the quiver Q of affine type $\tilde{A}_{4,4}$:



For any $i \in \mathbb{Z}/8\mathbb{Z}$, denote by $E_{i,i+1}$ the quasi-simple module whose composition factors are S_i and S_{i+1} . Then $\tau E_{i,i+1} = E_{i+2,i+3}$ for all $i \in \mathbb{Z}/8\mathbb{Z}$. As usual, we

denote by $E_{ij}^{(n)}$ the unique indecomposable regular module with quasi-length n and quasi-socle E_{ij} .

The AR-quiver of $kQ\text{-mod}$ contains exactly two exceptional tubes which are of rank $p = 4$. We consider the tube depicted in figure 3.

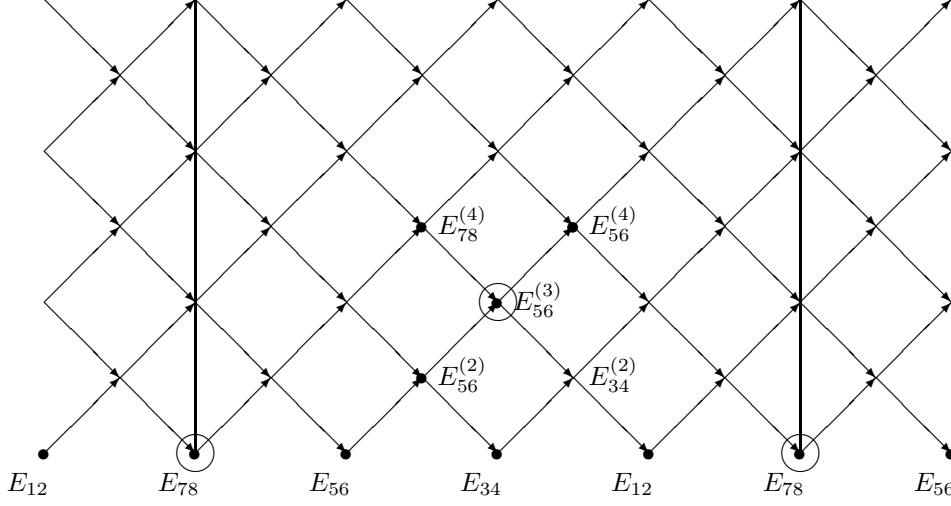


FIGURE 3. An exceptional tube of $\tilde{\mathbb{A}}_{4,4}$

We consider the module $M = E_{78} \oplus E_{56}^{(3)}$. The multiplication formula gives

$$X_M = X_{E_{56}^{(4)}} + X_{E_{56}^{(2)}}$$

but according to the difference property proved in theorem 3.25, we have

$$X_{E_{56}^{(4)}} = X_{M_\lambda} + X_{E_{34}^{(2)}}$$

for every $\lambda \in \mathbb{P}_0^1(Q)$. As $E_{34}^{(2)}$ and $E_{56}^{(2)}$ are indecomposable rigid, they belong to $\mathcal{B}'(Q)$. Moreover $X_{M_\lambda} = X_\delta$ so

$$\begin{aligned} X_M &= X_{M_\lambda} + X_{E_{34}^{(2)}} + X_{E_{56}^{(2)}} \\ &= X_\delta + X_{\dim E_{34}^{(2)}} + X_{\dim E_{56}^{(2)}} \\ &\in \mathbb{Z}\mathcal{B}'(Q). \end{aligned}$$

5. EXAMPLES OF SEMICANONICAL BASES

5.1. Semicanonical basis for Dynkin quivers. In this short subsection, we prove that the generic variables are the elements of the Caldero-Keller basis found in [CK08] for a quiver of Dynkin type.

Proposition 5.1. *Let Q be a Dynkin quiver. Then*

$$\mathcal{B}'(Q) = \{\text{cluster monomials in } \mathcal{A}(Q)\}$$

is a \mathbb{Z} -basis for the \mathbb{Z} -module $\mathcal{A}(Q)$.

Proof. We denote by $\mathcal{M}(Q) = \{\text{cluster monomials in } \mathcal{A}(Q)\}$. It follows from lemma 3.10 that $\mathcal{M}(Q) \subset \mathcal{B}'(Q)$. Fix now an element $\mathbf{d} \in \mathbb{N}^{Q_0}$ and write $\mathbf{d} = \bigoplus_{i=1}^n \mathbf{e}_i$ its canonical decomposition. Then, according to proposition 3.5, for every $i = 1, \dots, n$,

there is a Schur representation $M_i \in \text{rep}(Q, \mathbf{e}_i)$ such that $\text{Ext}_{kQ}^1(M_i, M_j) = 0$ for $i \neq j$. As Q is Dynkin, every root is a real root and then each \mathbf{e}_i is a real Schur root and thus each M_i is rigid. It follows that $M = \bigoplus_{i=1}^n M_i$ is rigid in $\text{rep}(Q, \mathbf{d})$ and then $\mathcal{O}_M \cap U_{\mathbf{d}} \neq \emptyset$. It follows that $X_{\mathbf{d}} = X_M$ for some rigid module M and thus $X_{\mathbf{d}}$ is a cluster monomial.

Assume now that \mathbf{d} is any element in \mathbb{Z}^{Q_0} . Then

$$X_{\mathbf{d}} = X_{[\mathbf{d}]_+} \prod_{d_i < 0} u_i^{-d_i}.$$

By the above discussion, there exists some rigid module $M \in \text{rep}(Q, [\mathbf{d}]_+)$ such that $X_M = X_{[\mathbf{d}]_+}$. Moreover, for every $i \in Q_0$ such that $d_i < 0$, we have $\text{Ext}_{\mathcal{C}}^1(M, P_i[1]) = 0$ so $N = M \oplus \bigoplus_{d_i < 0} P_i[1]^{\oplus (-d_i)}$ is rigid in \mathcal{C}_Q and $X_{\mathbf{d}} = X_N$ is a cluster monomial.

This is a \mathbb{Z} -basis according to [CK08]. \square

5.2. Bases for the Kronecker quiver. The first example of cluster algebra of non Dynkin type that has already been studied is the most simple case of affine quiver, namely the Kronecker quiver

$$K : 1 \rightleftarrows 2.$$

We write $\delta = (1, 1)$ the minimal imaginary root of K and we denote by $\mathcal{A}(K)$ the coefficient-free cluster algebra with initial seed (K, \mathbf{u}) with $\mathbf{u} = (u_1, u_2)$.

5.2.1. Normalized Chebyshev polynomials. In this subsection, we recall briefly some results of [CZ06] concerning the normalized Chebyshev polynomials.

The *normalized Chebyshev polynomials of the second kind* are the polynomials defined inductively by :

$$C_{-2}(x) = 0, C_{-1}(x) = 0, C_0(x) = 1, C_{n+1}(x) = xC_n(x) - C_{n-1}(x)$$

In particular, for every $n \geq 0$, C_n is the unitary polynomial of degree n characterized by the identity

$$C_n(t + t^{-1}) = \sum_{k=0}^n t^{n-2k}$$

The *normalized Chebyshev polynomials of the first kind* are the polynomials defined inductively by :

$$P_n(x) = C_n(x) - C_{n-2}(x)$$

for every $n \geq 0$. In particular, P_n is also a unitary polynomial of degree n for any $n \geq 0$ satisfying

$$P_n(t + t^{-1}) = t^n + t^{-n}$$

We can express the C_n in terms of P_n with the relation

$$C_n(x) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} P_{n-2k}.$$

In particular C_n is a positive linear combination of the P_k .

5.2.2. The canonical basis for the Kronecker quiver. We recall briefly the results and notations of [SZ04].

Definition 5.2. An element $y \in \mathcal{A}(K)$ is called positive if for every cluster $\mathbf{x} = (x, x')$, the coefficients in the expansion of y as a Laurent polynomial in x and x' are positive.

Theorem 5.3 ([SZ04]). *There exists a unique \mathbb{Z} -basis $\mathcal{B}(K)$ of $\mathcal{A}(K)$ such that the semi-ring of positive elements in $\mathcal{A}(K)$ consists precisely of positive integer linear combinations of elements of $\mathcal{B}(K)$.*

The set $\mathcal{B}(K)$ is called the *canonical basis* of the cluster algebra $\mathcal{A}(K)$. The explicit value of the canonical basis is explicitly computed in [SZ04] and is given by

$$\mathcal{B}(K) = \{\text{cluster monomials}\} \sqcup \{P_n(z) : n \geq 1\}$$

where

$$z = \frac{1 + u_1^2 + u_2^2}{u_1 u_2}.$$

It is proved in [SZ04] that for any $n \geq 1$, we have

$$\delta(P_n(z)) = n\delta$$

Thus the canonical basis is the disjoint union of the set of cluster variables and a family of variables whose denominator vector are the imaginary roots of Q .

5.2.3. The semi-canonical basis for the Kronecker quiver. We denote by $\mathcal{B}'(K)$ the semi-canonical basis of the cluster algebra associated to K . This is given by

$$\mathcal{B}'(K) = \{X_\alpha : \alpha \in \mathbb{Z}^{Q_0}\}$$

The explicit description is given by proposition 3.18 but we give an independent and explicit construction of all the $X_{\mathbf{d}}$ based on the known results about the canonical decomposition of elements in the root lattice of the Kronecker quiver.

Proposition 5.4.

$$\mathcal{B}'(K) = \{\text{cluster monomials}\} \sqcup \{z^n : n \geq 1\}$$

Proof. It follows from lemma 3.10 that cluster monomials are elements of the canonical basis. Also, a direct computation proves that

$$X_\delta = X_{M_\lambda} = z$$

for any homogeneous simple M_λ .

We also know that $X_{n\delta} = X_\delta^n = z^n$. Now we claim the above union is disjoint. In deed, consider an element z^n and a cluster monomial x . Then there is some M without self-extension such that $X_M = x$. Thus it follows that $\mathbf{dim} M \notin \mathbb{Z}_{\geq 0}\delta$ (there is no representation without self extension in $\text{rep}(Q, n\delta)$ for every n). Thus $\delta(x) \neq \delta(z^n)$ for every n and the above union is disjoint.

Now it only remains to prove that every element $X_{\mathbf{d}}$ is either a cluster monomial or a z^n for some $n > 0$. First assume that $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then we know that $X_{\mathbf{d}} = X_M$ for some M in the open set of Kac's canonical decomposition $\mathfrak{M}_{\mathbf{d}}$. We write $\mathbf{d} = (d_1, d_2)$. Then it follows from [DW02, subsection 3] that the canonical decomposition of \mathbf{d} depends on the quotient $a = d_1/d_2$. More precisely, if $a > 1$, then the canonical decomposition of \mathbf{d} is of the form $\lambda p_m + \mu p_{m+1}$ where $p_{2k+1} = \mathbf{dim} P_{2k+1} = \mathbf{dim} \tau^{-k} P_1$ and $p_{2(k+1)} = \mathbf{dim} P_{2(k+1)} = \mathbf{dim} \tau^{-k} P_2$. Thus

$$X_{\mathbf{d}} = X_{P_m^\lambda \oplus P_{m+1}^\mu}$$

but $\text{Ext}_{kQ}^1(P_m, P_{m+1}) = \text{Ext}_{kQ}^1(P_{m+1}, P_m) = 0$ the $X_{\mathbf{d}}$ is a cluster monomial. Similarly, if $a < 1$, $X_{\mathbf{d}}$ will be a cluster monomial. Now if $a = 1$, $\mathbf{d} = n\delta$ for some n and then we know that

$$X_{\mathbf{d}} = X_{n\delta} = X_\delta^n = X_{M_\lambda}^n = z^n$$

Now suppose that $\mathbf{d} \in \mathbb{Z}^{Q_0}$ is such that $d_i < 0$ for some i . If $d_1 < 0$ and $d_2 < 0$, then

$$X_{\mathbf{d}} = X_{P_1[1]^{\oplus(-d_1)} \oplus P_2[1]^{\oplus(-d_2)}}$$

is a cluster monomial because $P_1[1]^{\oplus(-d_1)} \oplus P_2[1]^{\oplus(-d_2)}$ has no self-extension.

If $d_1 < 0$ and $d_2 \geq 0$, then

$$X_{\mathbf{d}} = X_{P_1[1]^{\oplus(-d_1)} \oplus I_2^{\oplus d_2}}$$

is a cluster monomial because $P_1[1]^{\oplus(-d_1)} \oplus I_2^{\oplus d_2}$ has no self-extension.

Finally, if $d_1 \geq 0$ and $d_2 < 0$, then

$$X_{\mathbf{d}} = X_{P_1^{\oplus d_1} \oplus P_2[1]^{\oplus(-d_2)}}$$

is a cluster monomial because $P_1^{\oplus d_1} \oplus P_2[1]^{\oplus(-d_2)}$ has no self-extension.

Now it proves that

$$\mathcal{B}'(K) = \{\text{cluster monomials}\} \sqcup \{z^n : n \geq 1\}$$

□

Notice that $P_n(z) \neq z^n$ for every $n > 1$. It follows that the semi-canonical basis and the canonical basis do not coincide. Thus, the semi-canonical do not have the positivity property of theorem 5.3.

Nevertheless, for any $n \geq 1$, the denominator vector $\delta(z^n)$ of z^n is $n\delta(z) = n\delta$. Thus, as for the canonical basis, the semi-canonical basis is the disjoint union of the set of cluster variables and a family of variables whose denominator vector are the positive imaginary roots of Q .

5.2.4. *Caldero-Zelevinsky basis for the Kronecker quiver.* In [CZ06], the authors have computed another \mathbb{Z} -basis for the Kronecker quiver, this basis is given by

$$\mathcal{B}''(K) = \{\text{cluster monomials}\} \sqcup \left\{ X_{M_{\lambda}^{(n)}} : n \geq 1 \right\}$$

In corollary 3.13, we proved that $X_{M_{\lambda}^{(n)}} = C_n(z)$ where the C_n are the normalized Chebyshev polynomial of the second kind defined in subsection 5.2.1.

As it was noticed before, the denominator theorem of [CK06] implies that

$$\delta(C_n(z)) = \delta(X_{M_{\lambda}^{(n)}}) = n\delta.$$

Thus the basis $\mathcal{B}''(K)$ is also the disjoint union of the set of cluster monomials and of a sets of variables whose denominator vectors correspond to the positive imaginary roots of Q .

5.2.5. *Base change between $\mathcal{B}(K)$ and $\mathcal{B}'(K)$.*

Definition 5.5. Let $\mathbf{a} = \{a_n, n \geq 0\}$ and $\mathbf{b} = \{b_n, n \geq 0\}$ be two bases of the \mathbb{Z} -module $\mathcal{A}(K)$. We say that there is a *locally unipotent base change* from A to B if for every $n \in \mathbb{Z}$, the \mathbb{Z} -modules spanned by $\{a_k, 0 \leq k \leq n\}$ and $\{b_k, 0 \leq k \leq n\}$ coincide and if the base change matrix P from $(a_k, 0 \leq k \leq n)$ to $(b_k, 0 \leq k \leq n)$ is unipotent in $M_n(\mathbb{Z})$. If moreover P has positive entries, then the base change is called *positive*.

Proposition 5.6. *There is a positive locally unipotent base change from $\mathcal{B}(K)$ to $\mathcal{B}'(K)$.*

Proof. As

$$\mathcal{B}(K) = \{\text{cluster monomials}\} \sqcup \{P_n(z), n \in \mathbb{N}\}$$

and

$$\mathcal{B}'(K) = \{\text{cluster monomials}\} \sqcup \{z^n, n \in \mathbb{N}\},$$

it suffices to prove that there is a positive locally unipotent base change from $\{P_n(z), n \in \mathbb{N}\}$ to $\{z^n, n \in \mathbb{N}\}$. It is equivalent to prove that every z^n can be written as a positive \mathbb{Z} -linear combination of the $P_k(z)$ for $0 \leq k \leq n$.

As $\mathcal{B}(K)$ is a \mathbb{Z} -basis of $\mathcal{A}(K)$ and $z^n \in \mathcal{A}(K)$ for every n , it follows that each z^n can be written as a \mathbb{Z} -linear combination of $P_k(z)$ for $k \in \mathbb{N}$. Each $P_k(z)$ being a unitary polynomial of degree k , it follows that z^n can be written as a \mathbb{Z} -linear combination of the $P_k(z)$ for $0 \leq k \leq n$. Thus, there is a locally unipotent base change from $\mathcal{B}(K)$ to $\mathcal{B}'(K)$.

According to [SZ04], $z = X_{M_\lambda}$ is a positive element in $\mathcal{A}(K)$. Thus, as positive elements form a semiring in $\mathcal{A}(K)$, each $X_{M_\lambda}^n$ is a positive element in $\mathcal{A}(K)$ and can thus be written as a positive \mathbb{Z} -linear combination of elements of $\mathcal{B}(K)$. The base change is then positive and the proposition is proved. \square

Example 5.7. If we look at the base change matrix from the family $(z^n, 0 \leq n \leq 10) \subset \mathcal{B}'(K)$ to the family $(P_n(z), 0 \leq n \leq 10)$ of the canonical basis for $0 \leq n \leq 10$, we obtain

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 6 & 0 & 20 & 0 & 70 & 0 & 252 \\ 0 & 1 & 0 & 3 & 0 & 10 & 0 & 35 & 0 & 126 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 & 15 & 0 & 56 & 0 & 210 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 & 21 & 0 & 84 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 28 & 0 & 120 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 7 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 8 & 0 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which is positive and unipotent. The inverse matrix is

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 & -7 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 9 & 0 & -16 & 0 & 25 \\ 0 & 0 & 0 & 1 & 0 & -5 & 0 & 14 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 20 & 0 & -50 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -8 & 0 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2.6. *Base change between $\mathcal{B}'(K)$ and $\mathcal{B}''(K)$.* For any $n \geq 0$, we write $P_n = P_n(z)$ and

$$z^n = \sum_{i \leq n} \lambda_{i,n} P_i$$

the expansion of z^n in the P_n . It follows from proposition 5.6 that each $\lambda_{i,n}$ is positive.

Lemma 5.8. *For any $n \geq 1$, we have :*

- (1) $\lambda_{i,n} = 0$ if $i \not\equiv n[2]$,
- (2) $\lambda_{i,n} < \lambda_{i-2,n}$ for any $i \geq 2$ such that $i \equiv n[2]$.

Proof. We prove it by induction on n . If $n = 1$, then $z^n = P_1$ and thus $\lambda_{1,1} = 1$ and $\lambda_{i,1} = 0$ for all $i \neq 1$, the above assertions are true. We now prove the induction step.

$$z^{n+1} = z \cdot z^n = P_1 \cdot \left(\sum_{i \leq n} \lambda_{i,n} P_i \right)$$

$$P_1 P_i = \begin{cases} P_{i-1} + P_{i+1} & \text{if } i > 1 \\ 2 + P_2 & \text{if } i = 1 \\ P_1 & \text{if } i = 0 \end{cases}$$
$$\begin{aligned} z^{n+1} &= \lambda_{0,n}P_1P_0 + \lambda_{1,n}P_1P_1 + \sum_{2 \leq i \leq n} \lambda_{i,n}P_1P_i \\ &= \lambda_{0,n}P_1 + \lambda_{1,n}(2 + P_2) + \sum_{2 \leq i \leq n} \lambda_{i,n}(P_{i-1} + P_{i+1}) \\ &= 2\lambda_{1,n}P_0 + (\lambda_{0,n} + \lambda_{2,n})P_1 + \sum_{i \geq 2} (\lambda_{i-1,n} + \lambda_{i+1,n})P_i \end{aligned}$$
$$\mathcal{B}'(K) = \{\text{cluster monomials}\} \sqcup \{z^n, n \geq 0\}$$
$$\mathcal{B}''(K) = \{\text{cluster monomials}\} \sqcup \{C_n(z), n \geq 0\},$$

We denote by $C_n = C_n(z)$. Then we recall that $P_n = C_n - C_{n-2}$. We write

$$z^n = \sum_{i \leq n} \lambda_{i,n} P_n$$

$$\begin{aligned} z^n &= \sum_i \lambda_{i,n} (C_n - C_{n-2}) \\ &= \sum_i (\lambda_{i,n} - \lambda_{i+2,n}) C_n \end{aligned}$$
[illegible]

which is positive and unipotent. The inverse matrix is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 0 & 3 & 0 & -4 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 6 & 0 & -10 & 0 & 15 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 & 10 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 & 0 & 15 & 0 & -35 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 21 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5.3. Semicanonical basis for cluster algebras of type $\tilde{A}_{2,1}$. We are now interested in the particular case of quivers of affine type $\tilde{A}_{2,1}$, which is the only example of simply laced affine cluster algebra of rank 3. Such a quiver is necessarily isomorphic to

$$Q : \begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \xrightarrow{\quad} & 3 \end{array} .$$

For any $\lambda \in k$, we set M_λ to be the representation given by

$$M_\lambda : \begin{array}{ccc} & k & \\ \nearrow 1 & & \searrow \lambda \\ k & \xrightarrow{\quad 1 \quad} & k \end{array}$$

and we set

$$M_\infty : \begin{array}{ccc} & k & \\ \nearrow 1 & & \searrow 1 \\ k & \xrightarrow{\quad 0 \quad} & k \end{array} .$$

We identify \mathbb{P}^1 and $k \sqcup \{\infty\}$. For any $0 \neq \lambda \in \mathbb{P}^1 \setminus \{\infty\}$, M_λ is a quasi-module in an homogeneous tube and $\mathbb{P}_0^1 = \mathbb{P}^1 \setminus \{0\}$. The only exceptional tube is \mathcal{T}_0 of rank 2 whose quasi-simples are

$$E_0 : \begin{array}{ccc} & k & \\ \nearrow & & \searrow 0 \\ 0 & \xrightarrow{\quad} & 0 \end{array}$$

and

$$E_1 : \begin{array}{ccc} & 0 & \\ \nearrow 0 & & \searrow \\ k & \xrightarrow{\quad 1 \quad} & k \end{array} .$$

We denote by

$$\begin{aligned} x_0 &= X_{E_0} = \frac{u_1 + u_3}{u_2} \\ x_1 &= X_{E_1} = \frac{1 + u_1 u_2 + u_2 u_3}{u_1 u_3} \\ z &= X_{M_\lambda} = \frac{u_1^2 u_2 + u_1 + u_3 + u_2 u_3^2}{u_1 u_2 u_3 u_4} \end{aligned}$$

Note that for $i \neq j$, we have

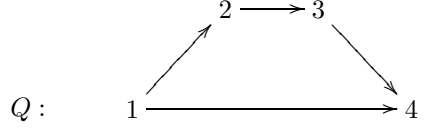
$$\text{Ext}_{\mathcal{C}_Q}^1(E_i, E_j) \neq 0,$$

thus, the only rigid modules in the additive closure of $\{E_i : i = 0, 1\}$ are $E_0^{\oplus n}$ and $E_1^{\oplus n}$ for $n \geq 1$.

According to proposition 3.18, we have then

$$\mathcal{B}'(Q) = \{\text{cluster monomials}\} \sqcup \{z^n x_i^r : n > 0, r \geq 0, i = 0, 1\}.$$

5.4. Semicanonical basis for cluster algebras of type $\tilde{\mathbb{A}}_{3,1}$. We are now interested in a slightly more complicated example that we already met in example 3.30. Consider a quiver Q of type $\tilde{\mathbb{A}}_{3,1}$, it is necessarily isomorphic to the quiver



We keep the notations of example 3.30. We compute

$$\begin{aligned} x_0 &= X_{E_0} = \frac{u_2 + u_4}{u_3} \\ x_1 &= X_{E_1} = \frac{u_1 + u_3}{u_2} \\ x_2 &= X_{E_2} = \frac{1 + u_1 u_3 + u_2 u_4}{u_1 u_4} \\ y_0 &= X_{E_0^{(2)}} = \frac{u_2 u_1 + u_1 u_4 + u_4 u_3}{u_2 u_3} \\ y_1 &= X_{E_1^{(2)}} = \frac{u_3 u_1^2 + u_1 + u_3^2 u_1 + u_3 + u_3 u_2 u_4}{u_1 u_2 u_4} \\ y_2 &= X_{E_2^{(2)}} = \frac{u_2 u_3 u_1 + u_2 + u_2^2 u_4 + u_4 + u_2 u_4^2}{u_1 u_3 u_4} \\ z &= X_{M_\lambda} = \frac{u_3 u_2 u_1^2 + u_2 u_1 + u_1 u_4 + u_4 u_3 + u_3 u_2 u_4^2}{u_1 u_2 u_3 u_4} \end{aligned}$$

Note that for $i \neq j$, we have

$$\text{Ext}_{\mathcal{C}_Q}^1(E_i, E_j) \neq 0,$$

$$\text{Ext}_{\mathcal{C}_Q}^1(E_i^{(2)}, E_j^{(2)}) \neq 0,$$

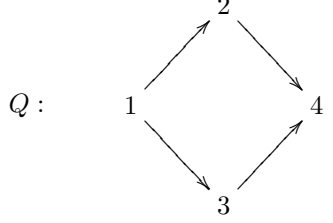
$$\text{Ext}_{\mathcal{C}_Q}^1(E_i, E_j^{(2)}) \neq 0,$$

Finally, for any $i = 0, 1, 2$, we have $\text{Ext}_{\mathcal{C}_Q}^1(E_i, E_i^{(2)}) = 0$. Thus, the only rigid modules in the additive closure of $\{E_i, E_i^{(2)} : i = 0, 1, 2\}$ are the $E_i^{\oplus r} \oplus (E_i^{(2)})^{\oplus s}$ for $r, s \geq 0$ and $i = 0, 1, 2$.

According to proposition 3.18, we have then

$$\mathcal{B}'(Q) = \{\text{cluster monomials}\} \sqcup \{z^n x_i^r y_i^s : n > 0, r, s \geq 0, i = 0, 1, 2\}.$$

5.5. Semicanonical basis for cluster algebras of type $\tilde{A}_{2,2}$. The only other example of (simply laced) rank 4 affine cluster algebra is the affine type $\tilde{A}_{2,2}$. Up to one mutation, we can assume that the considered quiver is



The AR-quiver of Q contains two exceptional tubes of rank 2 \mathcal{T}_0 and \mathcal{T}_∞ . The quasi-simples of \mathcal{T}_0 are denoted by E_0 and E_1 and the quasi-simples of \mathcal{T}_∞ are denoted by F_0 and F_1 .

We compute

$$\begin{aligned}
 x_0 &= X_{E_0} = \frac{u_1 + u_4}{u_2} \\
 x_1 &= X_{E_1} = \frac{u_1 + u_4 + u_1 u_2 u_3 + u_2 u_3 u_4}{u_1 u_3 u_4} \\
 y_0 &= X_{F_0} = \frac{u_1 + u_4}{u_3} \\
 y_1 &= X_{F_2} = \frac{u_1 + u_4 + u_1 u_2 u_3 + u_2 u_3 u_4}{u_1 u_2 u_4} \\
 z &= X_{M_\lambda} = \frac{u_1^2 + u_4^2 + 2u_1 u_4 + u_1^2 u_2 u_3 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}
 \end{aligned}$$

Note that, we have

$$\text{Ext}_{\mathcal{C}_Q}^1(E_0, E_1) \simeq k^2 \simeq \text{Ext}_{\mathcal{C}_Q}^1(F_0, F_1),$$

$$\text{Ext}_{\mathcal{C}_Q}^1(E_i, F_j) = 0 \text{ for } i \neq j.$$

Thus, the only rigid modules in the additive closure of $\{E_0, E_1, F_0, F_1\}$ are the $E_i^{\oplus r} \oplus F_j^{\oplus s}$ for $r, s \geq 0$ and $i, j = 0, 1$.

According to proposition 3.18, we have then

$$\mathcal{B}'(Q) = \{\text{cluster monomials}\} \sqcup \{z^n x_i^r y_j^s : n > 0, r, s \geq 0, i, j = 0, 1\}.$$

6. CONJECTURES AND QUESTIONS

6.1. Semicanonical basis for cluster algebras of affine types \tilde{E}, \tilde{D} . We conjecture that corollary 4.21 holds for any quiver of affine type.

Conjecture 6.1. *Let Q be an affine quiver. Then the set $\mathcal{B}'(Q)$ of all generic variables is a \mathbb{Z} -basis in $\mathcal{A}(Q)$.*

In order to prove this conjecture, it follows from theorem 4.20 that one only has to prove that a quiver of affine type \tilde{D} or \tilde{E} satisfies the difference property.

It seems unlikely to carry out proposition 3.23 to the other affine types. The possible multiplicities in the grassmannians do not leave any hope to adapt directly the proof given in this article. Using the notations of proposition 3.23, the first conjecture is:

Conjecture 6.2. *Let Q be an affine quiver of affine type, $\lambda \in \mathbb{P}_0^1(Q)$ and E be a quasi-simple module in an exceptional tube. Then, for any dimension vector \mathbf{v} , the following equality holds:*

$$\chi(\mathrm{Gr}_{\mathbf{v}}(M_E)) = \chi(\mathrm{Gr}_{\mathbf{v}}(M_\lambda)) + \chi(\mathrm{Gr}_{\mathbf{v}}^E(\mathrm{q.rad} M_E))$$

An unsatisfying method, but at least fruitful in simple examples, consists in doing explicit computations of cluster variables. In order to have a finite number of computations, one can chose a given orientation and try to deal later with reflection functors in order to obtain the result for any orientations. Nevertheless, it seems rather hopeless to do the computations for affine types $\tilde{\mathbb{E}}$.

As proposition 3.23 is only used in order to prove theorem 3.25, one can also try to prove directly theorem 3.25. The following conjecture would be a direct consequence of conjecture 6.2 but it might be simpler to prove it directly.

Conjecture 6.3. *Every quiver of affine type satisfies the difference property.*

The following lemma can turn out to be very useful, it proves that the difference $X_{M_E} - X_{\mathrm{q.rad} M_E/E}$ is invariant under translation. Namely:

Lemma 6.4. *Let Q be a quiver of affine type, let E be a quasi-simple module in an exceptional tube \mathcal{T} . Then*

$$X_{M_E} - X_{\mathrm{q.rad} M_E/E} = X_{M_F} - X_{\mathrm{q.rad} M_F/F}$$

for any quasi-simple module F in \mathcal{T} .

Proof. Fix Q an affine quiver, E a quasi-simple in an exceptional tube \mathcal{T} of rank $p > 1$. Denote by $E_i(Q), i \in \mathbb{Z}/p\mathbb{Z}$ the quasi-simple modules in \mathcal{T} such that $\tau E_i(Q) = E_{i-1}(Q)$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We denote by $X_{\mathcal{T}}^Q$ the Caldero-Chapoton map on the cluster category \mathcal{C}_Q .

Consider the quiver A of affine type $\tilde{\mathbb{A}}_{p,,p}$ with an alternating orientation. Fix $\mathcal{T}(A)$ an exceptional tube of $kA\text{-mod}$. Then $\mathcal{T}(A)$ has rank p . Denote by $E_i(A), i \in \mathbb{Z}/p\mathbb{Z}$ the quasi-simple modules in \mathcal{T} such that $\tau E_i(A) = E_{i-1}(A)$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We denote by $X_{\mathcal{T}}^A$ the Caldero-Chapoton map on the cluster category \mathcal{C}_A . It is proved in [Dup09] that the family $\{X_{E_i(A)}^A, i \in \mathbb{Z}/p\mathbb{Z}\}$ is algebraically independent over \mathbb{Q} .

We consider the surjective \mathbb{Z} -algebra homomorphism:

$$\pi : \begin{cases} \mathbb{Z}[X_{E_i(A)}^A, i \in \mathbb{Z}/p\mathbb{Z}] & \longrightarrow & \mathbb{Z}[X_{E_i(Q)}^Q, i \in \mathbb{Z}/p\mathbb{Z}] \\ X_{E_i(A)}^A & \mapsto & X_{E_i(Q)}^Q \end{cases} \quad \text{for any } i \in \mathbb{Z}/p\mathbb{Z}$$

It follows from [Dup09] that

$$\pi(X_{E_i(A)}^{A(k)}) = X_{E_i(Q)}^{Q(k)}$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$ and any $k \geq 0$.

In order to prove the lemma, it is equivalent to prove that

$$X_{E_i(Q)}^{Q(p)} - X_{E_{i-1}(Q)}^{Q(p-1)}$$

does not depend on $i \in \mathbb{Z}/p\mathbb{Z}$.

By theorem 3.25,

$$X_{E_i(A)}^{A(p)} - X_{E_{i-1}(A)}^{A(p-1)} = X_{M_\lambda}^A$$

where $\lambda \in \mathbb{P}_0^1(Q)$. In particular, the difference does not depend on $i \in \mathbb{Z}/p\mathbb{Z}$. Thus

$$X_{E_i(Q)}^{Q(p)} - X_{E_{i-1}(Q)}^{Q(p-1)} = \pi(X_{E_i(A)}^{A(p)} - X_{E_{i-1}(A)}^{A(p-1)})$$

does not depend on $i \in \mathbb{Z}/p\mathbb{Z}$. \square

This lemma finds its interest in the fact that for an affine quiver Q , Crawley-Boevey’s construction by one point extensions allows to realize an indecomposable module of dimension δ in each tube. This way, in any exceptional tube, there is a “privileged” module M_E , explicitly realized, for which it seems to be reasonable to prove equality (4). Lemma 6.4 can then prove that equality (4) holds for any indecomposable in the tube.

6.2. Semicanonical basis for acyclic quivers. For quivers of wild type, it is not clear that our methods can generalize. Indeed, our work involve deeply the knowledge of the Auslander-Reiten combinatorics of the considered quiver and at this time this Auslander-Reiten combinatorics is not so clear for wild quivers. Nevertheless, the very strong analogy between our works and those of Geiss, Leclerc and Schroër (see [GLS08]) about nilpotent varieties, there is a hope that the answer to the following question is positive:

Question 6.5. *Let Q be an acyclic quiver, is $\mathcal{B}'(Q) \cap \mathcal{A}(Q)$ a \mathbb{Z} -basis of the \mathbb{Z} -module $\mathcal{A}(Q)$?*

Acknowledgements. This work is part of my PhD thesis [Dup08]. I would like to thank my adviser Philippe Caldero for very helpful conversations and fruitful ideas about the subject. I would also like to thank Bernhard Keller and Robert Marsh for their careful reading through of this work and their various remarks and corrections. I am also grateful to Michel Brion for indicating the proof of lemma 3.1. Finally, I would like to thank Idun Reiten and Claus Michael Ringel for interesting discussions and advices concerning the topic.

REFERENCES

- [ASS05] I. Assem, D. Simson, and A. Skowroński. *Elements of representation theory of Associative Algebras, Volume 1: Techniques of Representation Theory*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, 2005. MR2197389 (2006j:16020).
- [BFZ05] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras III: Upper bounds and double Bruhat cells. *Duke Mathematical Journal*, 126(1):1–52, 2005. MR2110627 (2005i:16065).
- [BMR⁺06] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006. MR2249625 (2007f:16033).
- [BMR07] A. Buan, R. Marsh, and I. Reiten. Cluster-tilted algebras. *Trans. Amer. Math. Soc.*, 359(1):323–332, 2007. MR2247893 (2007f:16035).
- [BMR08] A. Buan, R. Marsh, and I. Reiten. Cluster mutation via quiver representations. *Commentarii Mathematici Helvetici*, 83(1):143–177, 2008.
- [BMR09] A. Buan, R. Marsh, and I. Reiten. Denominators of cluster variables. *J. Lond. Math. Soc. (2)*, 79(3):589–611, 2009.
- [CB92] W. Crawley-Boevey. Lectures on representations of quivers. 1992.
- [CBS02] W. Crawley-Boevey and J. Schröer. Irreducible components of varieties of modules. *J. Reine Angew. Math.*, 553:201–220, 2002. MR1944812 (2004a:16020).
- [CC06] P. Caldero and F. Chapoton. Cluster algebras as Hall algebras of quiver representations. *Commentarii Mathematici Helvetici*, 81:596–616, 2006. MR2250855 (2008b:16015).
- [CCS06a] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations and cluster tilted algebras. *Algebras and Representation Theory*, 9:359–376, 2006. MR2250652 (2007f:16036).
- [CCS06b] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations arising from clusters (A_n case). *Transactions of the AMS*, 358:1347–1354, 2006. MR2187656 (2007a:16025).
- [Cer08] G. Cerulli Irelli. *Structure theory for affine cluster algebras of rank 3*. PhD thesis, University of Padova, 2008.
- [CK06] P. Caldero and B. Keller. From triangulated categories to cluster algebras II. *Annales Scientifiques de l’Ecole Normale Supérieure*, 39(4):83–100, 2006. MR2316979 (2008m:16031).

- [CK08] P. Caldero and B. Keller. From triangulated categories to cluster algebras. *Inventiones Mathematicae*, 172:169–211, 2008. MR2385670.
- [CR08] P. Caldero and M. Reineke. On the quiver grassmannian in the acyclic case. *Journal of Pure and Applied Algebra*, 212(11):2369–2380, 2008. MR2440252.
- [CZ06] P. Caldero and A. Zelevinsky. Laurent expansions in cluster algebras via quiver representations. *Moscow Mathematical Journal*, 6:411–429, 2006. MR2274858 (2008j:16045).
- [DR76] V. Dlab and C.M. Ringel. Indecomposable representations of graphs and algebras. *Memoirs of the AMS*, 173:1–57, 1976. MR0447344.
- [Dup08] G. Dupont. *Algèbres amassées affines*. PhD thesis, Université Claude Bernard Lyon 1, <http://tel.archives-ouvertes.fr/tel-00338684/fr>, november 2008.
- [Dup09] G. Dupont. Cluster multiplication in regular components via generalized Chebyshev polynomials. *arXiv:0801.3964v2 [math.RT]*, 2009.
- [DW02] H. Derksen and J. Weyman. On the canonical decomposition of quiver representations. *Compositio Mathematica*, 133:245–265, 2002. MR1930979 (2003h:16017).
- [FZ02] S. Fomin and A. Zelevinsky. Cluster algebras I: Foundations. *J. Amer. Math. Soc.*, 15:497–529, 2002. MR1887642 (2003f:16050).
- [FZ03] S. Fomin and A. Zelevinsky. Cluster algebras II: Finite type classification. *Inventiones Mathematicae*, 154:63–121, 2003. MR2004457 (2004m:17011).
- [FZ07] S. Fomin and A. Zelevinsky. Cluster algebras IV: Coefficients. *Composition Mathematica*, 143(1):112–164, 2007. MR2295199 (2008d:16049).
- [GLS05] C. Geiss, B. Leclerc, and J. Schröer. Semicanonical bases and preprojective algebras. *Ann. Sci. École Norm. Sup. (4)*, 38(2):193–253, 2005.
- [GLS07] C. Geiss, B. Leclerc, and J. Schröer. Semicanonical bases and preprojective algebras II: A multiplication formula. *Compos. Math.*, 143(5):1313–1334, 2007. MR2360317 (2009b:17031).
- [GLS08] C. Geiss, B. Leclerc, and J. Schröer. Cluster algebra structures and semicanonical bases for unipotent groups. *arXiv:math/0703039v3 [math.RT]*, 2008.
- [HU05] D. Happel and L. Unger. On the set of tilting objects in hereditary categories. *Fields Institute Communications*, 45:141–159, 2005. MR2146246 (2006h:18006).
- [Hub06] A. Hubery. Acyclic cluster algebras via Ringel-Hall algebras. *preprint*, 2006.
- [Kac80] V.G. Kac. Infinite root systems, representations of graphs and invariant theory. *Inventiones Mathematicae*, 56:57–92, 1980. MR0557581 (82j:16050).
- [Kac82] V.G. Kac. Infinite root systems, representations of graphs and invariant theory II. *Journal of algebra*, 78:163–180, 1982. MR0677715 (85b:17003).
- [Kel05] B. Keller. On triangulated orbit categories. *Documenta Mathematica*, 10:551–581, 2005. MR2184464 (2007c:18006).
- [Kel08] B. Keller. Categorification of acyclic cluster algebras: an introduction. *arXiv:0801.3103v1 [math.RT]*, 2008.
- [KR08] B. Keller and I. Reiten. Acyclic calabi-yau categories. *Compositio Mathematicae*, 144(5):1332–1348., 2008. MR2457529.
- [Lec03] B. Leclerc. Imaginary vectors in the dual canonical basis of $U_q(n)$. *Transform. Groups*, 8(1):95–104, 2003. MR1959765 (2004d:17020).
- [MRZ03] R. Marsh, M. Reineke, and A. Zelevinsky. Generalized associahedra via quiver representations. *Trans. Amer. Math. Soc.*, 355(1):4171–4186, 2003. MR1990581 (2004g:52014).
- [Pal08] Y. Palu. Cluster characters for 2-Calabi-Yau triangulated categories. *Ann. Inst. Fourier (Grenoble)*, 58(6):2221–2248, 2008.
- [Rin84] C.M. Ringel. Tame algebras and integral quadratic forms. *Lecture Notes in Mathematics*, 1099:1–376, 1984. MR0774589 (87f:16027).
- [Sch92] A. Schofield. General representations of quivers. *Proc. London Math. Soc.*, 65(3):46–64, 1992. MR1162487 (93d:16014).
- [SZ04] P. Sherman and A. Zelevinsky. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Mosc. Math. J.*, 4:947–974, 2004. MR2124174 (2006c:16052).
- [Ver76] J.L. Verdier. Stratification de Whitney et théorème de Bertini-Sard. *Inventiones Mathematicae*, 36:295–312, 1976. MR0481096.
- [Xu10] F. Xu. On the cluster multiplication theorem for acyclic cluster algebras. *Trans. Amer. Math. Soc.*, 362(2):753–776, 2010.
- [XX07] J. Xiao and F. Xu. Green’s formula with \mathbb{C}^* -action and Caldero-Keller’s formula for cluster algebras. *arXiv:0707.1175v2 [math.QA]*, 2007.
- [Zhu06] B. Zhu. Equivalence between cluster categories. *Journal of Algebra*, 304:832–850, 2006. MR2264281 (2007h:18017).

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN CNRS UMR 5208,
43, BOULEVARD DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX.
E-mail address: `dupont@math.univ-lyon1.fr`